BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY

A Historical and Critical Review of Mathematical Science

EDITED BY

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VOL. XII

October 1905 to July 1906

PUBLISHED FOR THE SOCIETY BY
THE MACMILLAN COMPANY
Lancaster, Pa., and New York
1906

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Whole No. 141

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BULLETIN OF THE

AMERICAN MATHEMATICAL SOCIETY.

THE ELEMENTARY TREATMENT OF CONICS BY MEANS OF THE REGULUS.

BY PROFESSOR CHARLOTTE ANGAS SCOTT.

(Read before the American Mathematical Society, February 25, 1905.)

INASMUCH as projective plane geometry depends on properties of space (since the theorem on triangles in perspective in a plane cannot be established by plane geometry without the use of metric concepts), there is no logical objection to the employment of space considerations in proving theorems of plane geometry. The regulus supplies extremely simple proofs of the properties of a conic, whether this be considered as a locus or an envelope; these proofs have the advantage of connecting the points of a conic and the tangents of a conic from the first, instead of leaving the connection to be proved later by an elaborate chain of reasoning. The method makes unnecessary also the treatment of the tangent as the limiting position of a chord when the determining points become indistinguishable, which is open to some objections in pure geometry; not on account of the assumption as to continuity, for this has already been admitted in the proof of the fundamental theorem of projective geometry, but with regard to elegance and directness of proof. This note contains the application of the method to the proofs of the theorems of Chasles, Brianchon, and Pascal, and of polar and involution properties.

The regulus is the system of lines (rays) that meet three non-incident lines, the directors. By means of the fundamental theorem of projective geometry it is proved that the regulus is crossed by a second regulus; any three rays of either serve as directors of the other; through any point on a ray of either there passes a ray of the other.

Three directors, u, v, w, and three rays a, b, c (that is, two incident triads of lines) give rise to an interesting figure on which the proofs depend. The pairs of lines au, bv, cw determine points A, B, C and planes α, β, γ; the three planes meet in a point O, the three points lie in a plane ω. The point $\left(\begin{smallmatrix} a & b & c \\ u & v & w \end{smallmatrix}\right)$ and the plane $\left(\begin{smallmatrix} a & b & c \\ u & v & w \end{smallmatrix}\right)$, that is, O and ω, are pole and polar. The pairs of points (bw, cv), (cu, aw), (av, bu), that is, $\begin{smallmatrix} b & c \\ \times & \\ v & w \end{smallmatrix}$, $\begin{smallmatrix} c & a \\ \times & \\ w & u \end{smallmatrix}$, $\begin{smallmatrix} a & b \\ \times & \\ u & v \end{smallmatrix}$, lie on lines through O, namely, the lines $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$; the pairs of planes $\begin{smallmatrix} b & c \\ \times & \\ v & w \end{smallmatrix}$, $\begin{smallmatrix} c & a \\ \times & \\ w & u \end{smallmatrix}$, $\begin{smallmatrix} a & b \\ \times & \\ u & v \end{smallmatrix}$ meet in lines in ω, namely, BC, CA, AB. The points are named as shown in Fig. 1, the planes are named to correspond, *e. g.*, the point av is A', the plane av is α'; the planes α', β', γ' determine the point O', etc.

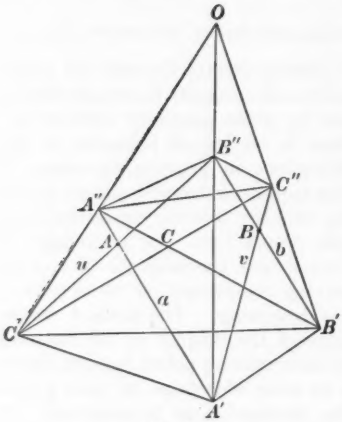

FIG. 1.

The triangles $A'B'C'$, $B''C''A''$ are perspective from O; $A''B''C''$, BCA from O'; ABC, $B'C'A'$ from O''. It is easily shown that O, O', O'' are collinear, $\omega, \omega', \omega''$ are coaxial; and that if the rays and directors are associated in the reversed order $\left(\begin{smallmatrix} a & c & b \\ u & v & w \end{smallmatrix}\right)$, the three new centres of perspective lie on the axis of $\omega\ \omega'\ \omega''$, and the axis of the new planes is $O\ O'\ O''$. These facts, however, are not needed for the present purpose.

It is at once seen that O, ω are harmonic to the pairs of lines au, bv, cw, that is, to the two triads of rays a, b, c and directors u, v, w, hence to the complete system of transversals to a, b, c (directors) and transversals to u, v, w (rays). This shows that the rays and directors intersect in pairs at points on ω, lie in pairs (the same pairs) on planes through O; the two of a pair d, x thus associated are harmonic with regard to O and ω. Any

chord through O joins a point on a ray d to a point on the associated director x, and is therefore divided harmonically by O, ω. Moreover, if O be any point on a chord that meets a ray d and director x, then the point dx lies on the polar plane of O.

Any line p, not a director, that meets one ray meets one other, but no more. For let p pass through A (*i. e.*, au); the plane pu meets v, w, at V, W; the line VW is therefore a ray, and is met by p. The only possible exception arises when VW is itself the ray a; then the line p meets the regulus at the point A only (or, the line p meets only one ray of the regulus); p is a tangent line, and the plane α (*i. e.*, au), in which lie all tangent lines through the point A (*i. e.*, au), is a tangent plane. The facts proved above as to the relation of O, ω may be stated in the form: The points of contact of tangent planes from the point $\left(\begin{smallmatrix} a & b & c \\ u & v & w \end{smallmatrix}\right)$ lie on the plane $\left(\begin{smallmatrix} a & b & c \\ u & v & w \end{smallmatrix}\right)$.

In order that a tangent plane may contain a line q, it must contain any two points S, S' on the line; hence the point of contact must lie on both σ and σ', that is on a line q'; q, q' are conjugate lines. Hence if there is one tangent plane through q, there is precisely one other, unless q' is itself a ray or director, which happens only when q is a ray or director, and then q' coincides with q.

The ranges of points determined on two directors u, v by the rays $a, b, c, d, \cdots$ are sections of the axial pencil $x.abcd \cdots$ (where x is any other director) by the transversals u, v; hence they are projective. The regulus is therefore the system of lines that join corresponding points of projective ranges on two non-incident lines. Again, the axial pencils $u.abcd \cdots$, $v.abcd \cdots$ have a common section by the transversal x, hence they are projective. The regulus is therefore the system of lines determined by corresponding planes of two projective axial pencils whose axes are non-incident.

Let the regulus be cut by a plane ω; each ray is thus associated with a particular director; the two, d, x, meet in a point D on ω, and lie in a plane δ (a tangent plane) through O. The projective axial pencils $u.abcd \cdots$, $v.abcd \cdots$ are cut in projective flat pencils, centers A and B; hence the plane section of a regulus (which is a point system of the second order, since a line through A cuts one other ray) is the locus of the intersection of corresponding rays of two projective flat

pencils. Consider also the line system composed of the tangent lines in the plane ω; these are the intersections of the tangent planes au, bv, cw, dx, ey, etc. by ω, they are therefore the projections of rays $a, b, c, d, e, \cdots$ (or of directors $u, v, w, x, y, \cdots$) from O on ω; call them $a', b', c', \cdots$. Since the ranges $u.abcd\cdots$, $v.abcd\cdots$ are projective, their projections from O on ω are projective. Hence the lines $a', b', c', d', \cdots$ in the plane ω connect the corresponding points of projective ranges on the two tangents a', b'; that is, the line system (which, by what has been said about tangent planes, is seen to be of the second order) is composed of the lines that join corresponding points of projective ranges. Moreover, as to the relation of the two systems: the flat pencil $A.BCD\cdots$ in the plane ω = the axial pencil $u.abcd\cdots$ = range $v.abcd\cdots$ = range $v'.a'b'c'd'\cdots$ in the plane ω. Thus the section (aggregate of points and lines) has the property that the pencil subtended at any point A by the points C, D, E, F is projective with the range determined on any tangent b' by the tangents c', d', e', f', which is Chasles's theorem. The section will now be called a conic.

Since the pencil determined by four points is projectively

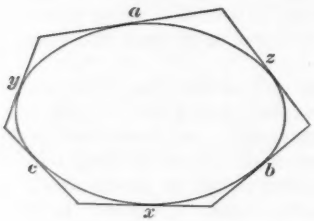

FIG. 2.

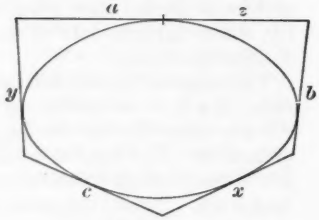

FIG. 3.

the same wherever on the section its vertex may lie, and similarly for the range determined by four tangents, it is unnecessary to specify the vertex or base; it is sufficient to speak of the four points or four lines. Since also the range determined by four rays on any director of the regulus depends only on the rays, it is sufficient to speak of the four rays. What has been proved above can be stated in the form: Four points of a conic are projective with the four rays, or four directors, that pass through them; four tangents to a conic are projective with the four rays, or four directors, that pass through their points of contact.

If four rays a, b, c, d are projective with four directors u, v, w, x, their intersections are coplanar. For let y be the director through the point D in which the plane au, bv, cw (that is, ABC) meets d; then the rays a, b, c, d are projective with the points A, B, C, D, and these are projective with the directors u, v, w, y. Hence $u\,v\,w\,x = u\,v\,w\,y$, and y is therefore the same as x.

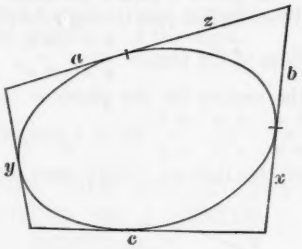

FIG. 4.

The polar properties follow at once from the harmonic relation borne to the rays and directors by O, ω. A point T in the plane ω has a polar plane τ, which by harmonic symmetry of the whole figure to $O\omega$ must pass through O. The section of τ by ω is the polar line of T with regard to the section considered, and it has already been shown that any chord through T, and therefore a chord of the section, is harmonically divided by $T\tau$. Hence follows the usual quadrilateral construction for pole and polar.

To prove Brianchon's theorem, take the six tangents as pro-

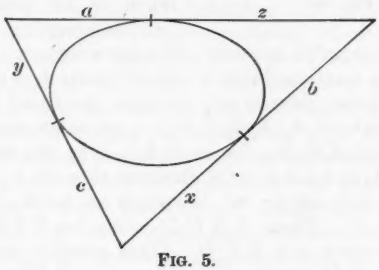

FIG. 5.

jections alternately of rays and directors, as shown in Fig. 2. In space, the lines $\overset{b\ c}{\underset{y\ z}{\times}}$, $\overset{c\ a}{\underset{z\ x}{\times}}$, $\overset{a\ b}{\underset{x\ y}{\times}}$ meet at a point S; projecting the figure from O on to ω, we obtain three concurrent lines joining opposite vertices of the circumscribing hexagon.

To prove the special cases of this theorem for the circum-

scribing pentagon, quadrilateral, triangle, regard the necessary number of tangents (one, two or three) as projections of both a ray and a director, as shown in Figs. 3, 4, 5; the proof then applies without change, the point az being the point of contact of the tangent.

To prove Pascal's theorem, take alternately the rays and directors that pass through the six points, Fig. 6. The common lines of the planes $\overset{b\ c}{\times}{}_{y\ z}$, $\overset{c\ a}{\times}{}_{z\ x}$, $\overset{a\ b}{\times}{}_{x\ y}$ lie in a plane σ; hence in the section by the plane ω the common points of the lines $\overset{b\ c}{\times}{}_{y\ z}$, $\overset{c\ a}{\times}{}_{z\ x}$, $\overset{a\ b}{\times}{}_{x\ y}$ lie in a line, the line $\sigma\omega$. The special cases of this theorem, which arise when sides of the inscribed hexagon (one, two, or three) are replaced by tangents, require no change in the proof. Moreover, the same proof applies when the section by the plane ω consists of a ray and a director, that is, when the theorem to be proved becomes Pappus's theorem, that if the vertices of a hexagon are taken to lie alternately on two intersecting lines, the three intersections of opposite sides are collinear.

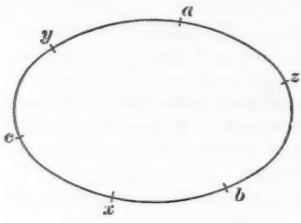

FIG. 6.

It has been mentioned that if a chord meets d, x, the point dx lies on the polar plane of any point on the chord. Hence if chords of a conic, A_1A_2, B_1B_2, C_1C_2, $\cdots$, are concurrent in S, then $A_1B_1C_1D_1 = A_2B_2C_2D_2$; for let a, b, c, d be the rays through A_1, B_1, C_1, D_1, and u, v, w, x the directors through A_2, B_2, C_2, D_2, and let the points au, bv, cw, dx (which are known to lie on σ) be U, V, W, X. Then $A_1B_1C_1D_1 = abcd = UVWX$ (in the section σ) $= uvwx = A_2B_2C_2D_2$. This proof is unaffected by coincidence; D_1D_2 may be the same as A_2A_1. Thus $A_1B_1C_1A_2 = A_2B_2C_2A_1$. Conversely, if in the plane ω $A_1B_1C_1A_2 = A_2B_2C_2A_1$, the chords A_1A_2, B_1B_2, C_1C_2 are concurrent. For since $A_1B_1C_1A_2 = abcd$, and $A_2B_2C_2A_1 = uvwx$, we have $abcd = uvwx$, and therefore by a result obtained earlier, the points au, bv, cw, dx lie on a plane σ. Hence the planes au, bv, cw, dx meet in a point S. But the planes au, dx meet in a line in the plane ω, hence the point S lies in ω. Since then the planes

au, *bv*, *cw*, *dx* meet at a point *S* in ω, the section of these planes by ω gives chords of the conic concurrent in *S*. Thus we have the theorem: If points of a conic are projectively paired, they lie on concurrent chords, which is the foundation of the theory of involution on a conic.

For the sake of brevity, as little detail as possible has been given in this note, the design being simply to draw attention to this mode of proving fundamental properties of the conic.

BRYN MAWR COLLEGE,
June, 1905.

ARZELÀ'S CONDITION FOR THE CONTINUITY OF A FUNCTION DEFINED BY A SERIES OF CONTINUOUS FUNCTIONS.

BY PROFESSOR E. J. TOWNSEND.

(Read before the American Mathematical Society, September 7, 1905.)

§ 1. A FUNCTION defined by a series whose terms are continuous functions may or may not be itself continuous. It may in fact be discontinuous at a set of points everywhere dense within the interval of definition. It is important to establish a criterion by which the continuity of such a function may be determined. Conditions which are sufficient, although not necessary, are to be found in any extensive work on calculus. Arzelà was the first to formulate a set of conditions which are both necessary and sufficient.* In his first discussion of the subject, however, he was not sufficiently rigorous.† A later and more rigorous development was given, differing from the first in some particulars.‡ Still more recently he has revised his first set of proofs and maintains that they are now sufficiently rigorous to be valid.§ It is the purpose of this paper to present in substance the final results of Arzelà's investigations.

*Intorno alla continuità della somma di infinite funzioni continue, Bologna, 1884.

†See Schoenflies, Punktmannigfaltigkeiten, p. 225, footnote. Also Arzelà, Sulla serie di funzioni di variabili reali, Bologna, 1902, p. 6.

‡Sulla serie di funzioni, part 1, Bologna, 1899, p. 10 et seq.

§See Sulla serie di funzioni di variabili reali, Bologna, 1902.

Let us then define $S_n(x)$ by the relation

$$S_n(x) = u_1(x) + u_2(x) + \cdots + u_\kappa(x) + \cdots + u_n(x), \quad a \leqq x \leqq \beta,$$

where $u_\kappa(x)$ and hence $S_n(x)$ is a continuous function of x. Moreover, let us define $f(x)$ by the equation

$$f(x) = \lim_{n=\infty} S_n(x), \qquad a \leqq x \leqq \beta.$$

The interval of convergence of the series is then the interval of definition of the function $f(x)$. The problem is to find a set of conditions which are necessary and sufficient for the continuity of $f(x)$ in a given interval (a, β), equal to or less than the interval of convergence.

With Arzelà, let us consider this problem as a special case of the following more general problem. Suppose $f(x, y)$ to be a function of the two independent variables x and y, defined for all values of x within the closed interval (a, β) and for

$$y \equiv (y) = y_1, y_2, y_3, \cdots, y_\nu, \cdots,$$

$y_1, y_2, y_3, \cdots, y_\nu, \cdots$ being a set of values dense at $y = y_0$, but not including y_0. Let $f(x, y_\nu)$ be a continuous function of x for each value y_ν included in the set (y). For $y = y_0$, let $f(x, y)$ be defined by the equation

$$f(x, y_0) = \lim_{y_\nu = y_0} f(x, y_\nu).$$

The more general problem, which we shall now consider, is to find a necessary and sufficient set of conditions that the limiting function $f(x, y_0)$ shall be a continuous function of x in the interval $a \leqq x \leqq \beta$. If we put $y_\nu = 1/n$, $n = 1, 2, 3, \cdots$, this reduces at once to the case of the infinite series given above.*

§ 2. First of all, let us consider the necessary and sufficient condition that the limiting function $f(x, y_0)$ shall be continuous at a single point x_0 of the interval (α, β). This condition may be stated as follows:

*The results given in this paper can be obtained, and perhaps more easily, without the introduction of the more general problem indicated. It is sufficient for this purpose to consider $S_n(x)$ as a function of the two variables x, n and to make of use the principles of ordinary convergence. Arzelà's method has been retained, not because it has any virtue in itself, but because it gives an opportunity to interpret more clearly his results at a point where he has been criticized.

I. *In order that the function $f(x, y_0)$, defined as in § 1 for the interval (α, β), be continuous at any point x_0 in this interval, it is necessary and sufficient that to an arbitrarily small positive number σ and to every y_t sufficiently near y_0 there shall correspond a neighbor. ood of the point x_0, which, however, may vary in extent with y_t, such that for every value of x in this neighborhood*

$$|f(x, y_0) - f(x, y_t)| < \sigma.$$

We shall first show that the above condition is necessary. Let us suppose then that $f(x, y_0)$ is a continuous function of x at the point $x = x_0$. By virtue of the limit

$$f(x_0, y_0) = \lim_{y_s = y_0} f(x_0, y_s),$$

there must exist a definite number y_{s_1} of the set (y) such that for every y_t of this set lying between y_{s_1} and y_0, we have

$$|f(x_0, y_0) - f(x_0, y_t)| < \sigma/3, \tag{1}$$

where σ as elsewhere in this paper is an arbitrarily small posi-

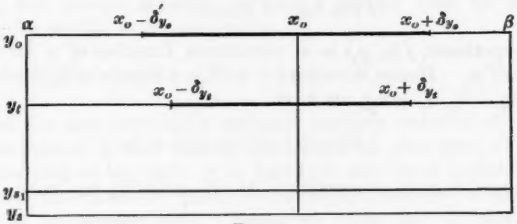

FIG. 1.

tive number. Because $f(x, y_0)$ is a continuous function of x for $y = y_0$, we have also for some neighborhood of x_0, say $(x_0 - \delta'_{y_0}, x_0 + \delta_{y_0})$, the inequality

$$|f(x, y_0) - f(x_0, y_0)| < \sigma/3. \tag{2}$$

Likewise, because $f(x, y_t)$ is a continuous function of x, there exists a neighborhood $(x_0 - \delta'_{y_t}, x_0 + \delta_{y_t})$ on the line $y = y_t$ (Fig. 1), such that for every x within it

$$|f(x_0, y_t) - f(x, y_t)| < \sigma/3. \tag{3}$$

Combining (1), (2), (3), we have

$$|f(x, y_0) - f(x, y_t)| < \sigma,$$

which holds for every y_t between y_{s_1} and y_0 and for every x within a neighborhood $(x_0 - d'_{y_t}, x_0 + d_{y_t})$, where d'_{y_t} is the smaller of the two values δ'_{y_0} and δ'_{y_t}, and d_{y_t} is the smaller of the two values δ_{y_0} and δ_{y_t}.

The given condition may be shown to be sufficient as follows. We have by hypothesis the inequality

$$|f(x, y_t) - f(x, y_0)| < \sigma, \tag{4}$$

which is valid for every value of x within a certain neighborhood, say $(x_0 - \delta', x_0 + \delta)$. The existence of the limit

$$f(x_0, y_0) = \lim_{y_t = y_0} f(x_0, y_t)$$

gives, for all values of y_t from a certain definite value on, the inequality

$$|f(x_0, y_0) - f(x_0, y_t)| < \sigma. \tag{5}$$

By hypothesis, $f(x, y_t)$ is a continuous function of x for each value of y_t. Hence, for every x within a certain neighborhood, say $(x_0 - \delta'_1, x_0 + \delta_1)$, we have

$$|f(x_0, y_t) - f(x, y_t)| < \sigma. \tag{6}$$

By combining the inequalities (4), (5), and (6), we have

$$|f(x_0, y_0) - f(x, y_0)| < 3\sigma,$$

which holds for all values of x lying within the smaller of the two neighborhoods $(x_0 - \delta', x_0 + \delta)$ and $(x_0 - \delta'_1, x_0 + \delta_1)$. This inequality establishes the continuity of $f(x, y_0)$ at the point $x = x_0$ as the theorem requires.

§ 3. Let us now consider the necessary and sufficient condition that $f(x, y_0)$ shall be a continuous function of x throughout the closed interval (α, β). This condition Arzelà states in substance as follows:

II. *In order that the function $f(x, y_0)$ defined as in § 1 be continuous throughout the interval $\alpha \leqq x \leqq \beta$, it is necessary and*

sufficient that to an arbitrarily small positive number σ and to an arbitrary $y_s \neq y_0$, there shall correspond a positive number l and a finite set of values

$$y = y_{s_1}, y_{s_2}, \cdots, y_{s_p}$$

lying between y_s and y_0, for which the following condition holds: On lines $y = y_{s_t} (t = 1, 2, \cdots, p)$, it shall be possible to choose a series of segments, each of length at least equal to l, whose projections on the line $y = y_0$, when taken together, completely cover the interval $\alpha \leqq x \leqq \beta$, and for which, furthermore, the relation

$$|f(x, y_0) - f(x, y_{s_t})| < \sigma$$

*holds, (x, y_{s_t}) being an arbitrary point on any one of these segments.**

It may be shown as follows that this condition is necessary.† Let us assume then that $f(x, y_0)$ is a continuous function of x throughout the interval $\alpha \leqq x \leqq \beta$. By virtue of theorem I, there exists for any $x' (\alpha \leqq x' \leqq \beta)$ and for each y_t sufficiently near y_0 a neighborhood $(x' - \delta'_{y_t}, x' + \delta_{y_t})$, which may vary with y_t but always, however, in such a manner that for each value of x within it the inequality

$$|f(x, y_0) - f(x, y_t)| < \sigma \tag{1}$$

is valid for any previously assigned positive value of σ. In this discussion Arzelà distinguishes between that part of the neighborhood to the right of x' and that part lying to the left. Let us denote these two parts of the neighborhood by $\Delta(x', y_t)$ and $\Delta'(x', y_t)$ respectively. These two parts may or may not be of the same magnitude. In other words, having chosen an arbitrarily small positive number σ and a y_s at pleasure, we may then select any value x' of the interval (α, β), end points included, and there will exist upon some line $y = y_t$ lying between $y = y_s$ and $y = y_0$ a neighborhood of magnitude $\Delta(x', y_t)$ to the right of x', such that for every x within it the inequality (1) is valid. Let us now consider $\Delta(x', y_t)$ for all values of y_t included in the set (y) and lying between y_0 and y_s. For some of these values of y_t, $\Delta(x', y_t)$ may be zero, and consequently

* In his statement of the theorem, Arzelà does not say that the given interval shall be closed. This, however, is necessary.

† See Sulle serie di funzioni, Bologna, 1899, p. 17, et seq.

for these particular values of y_t no neighborhood exists to the right of x' for which the above inequality holds. It may happen that for some values of y_t lying between y_0 and y_s this inequality does not hold even for x' itself. However $\Delta(x', y_t)$ cannot be zero for all values of y_t lying between the above limits unless x' is the extreme point β; for we know that when y_t is taken sufficiently near y_0 $\Delta(x', y_t)$ is always greater than zero. Having selected x', there may exist for any particular y_t a great variety of values which might be taken as the corresponding $\Delta(x', y_t)$. In what follows, let us understand by $\Delta(x', y_t)$ the upper limit of all these values. This magnitude $\Delta(x', y_t)$, regarded as a function of y_t for all values of y_t between y_s and y_0, has an upper limit greater than zero, providing x' is different from β. Such a superior limit is therefore definitely determined for each value of x', σ and y_s having been previously selected. Let us denote this upper limit by $\Delta(x')$. In the same way, consider that part of the interval $(x' - \delta'_{y_t}, x' + \delta_{y_t})$ lying to the left of x', and denote by $\Delta'(x')$ the corresponding upper limit, which must likewise be different from zero for x' different from α.

Consider now the sum

$$\Delta'(x') + \Delta(x').$$

This sum is uniquely determined for each value x' of x, where $\alpha \leqq x \leqq \beta$. As a function of x, it has a lower limit which we shall now show to be greater than zero. Denote this lower limit by l. There must exist then in (α, β) at least one point, in every neighborhood of which the lower limit of $\Delta'(x) + \Delta(x)$ is l. Let x_1 be such a point. For the point x_1 itself, we have the value $\Delta'(x_1) + \Delta(x_1)$, which is certainly greater than zero. Among the values of (y) included between y_s and y_0, there is at least one, say y_t, for which the magnitude $\Delta(x_1, y_t)$ of the neighborhood to the right of x_1, within which for each value of x we have

$$|f(x, y_0) - f(x, y_t)| < \sigma,$$

is as near $\Delta(x_1)$ as we may choose. In the same way, there exists a y_τ, which may be equal to or different from y_t, such that the magnitude $\Delta'(x_1, y_\tau)$ of the neighborhood to the left of x_1, within which we have for each value of x

$$|f(x, y_0) - f(x, y_\tau)| < \sigma,$$

is as near $\Delta'(x_1)$ as we choose to make it.

On the line $y = y_0$, consider the sub-interval

$$\left(x_1 - \frac{\Delta'(x_1, y_r)}{2}, x_1 + \frac{\Delta(x_1, y_t)}{2}\right).$$

For every value x'' included within this interval, there exists the neighborhood (Fig. 2)

$$\left(x_1, x_1 + \frac{\Delta(x_1, y_t)}{2}\right)$$

or the neighborhood

$$\left(x_1 - \frac{\Delta'(x_1, y_r)}{2}, x_1\right),$$

according as x'' is to the right or to the left of x_1, such that for

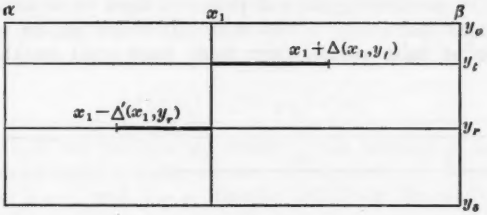

FIG. 2.

any value of x within these neighborhoods we have either

$$|f(x, y_0) - f(x, y_t)| < \sigma$$

or

$$|f(x, y_0) - f(x, y_r)| < \sigma.$$

Consequently, for every value x'' the corresponding value of $\Delta'(x) + \Delta(x)$ is greater than the smaller of the two numbers

$$\frac{\Delta'(x_1, y_r)}{2}, \qquad \frac{\Delta(x_1, y_t)}{2}.$$

The lower limit of $\Delta'(x) + \Delta(x)$, namely l, must then be greater than or at least equal to the smaller of the same two numbers. But we have already seen that these numbers can be made to differ as little as we choose from

$$\frac{\Delta'(x_1)}{2}, \qquad \frac{\Delta(x_1)}{2},$$

respectively, and these latter are certainly greater than zero. Hence we have in any case

$$l > 0.$$

Since the lower limit of $\Delta'(x) + \Delta(x)$ for $\alpha \leqq x \leqq \beta$ is greater than zero, it can be shown that a finite number of segments of the lines lying between $y = y_0$ and $y = y_s$ can be found fulfilling the requirements of the theorem. The projections of these segments upon $y = y_0$ may, and in general will, overlap. That a finite number of them is sufficient in order that the sum of their projections shall completely fill the interval (α, β) may be shown by considering the end points of those parts which the projections contribute to the sum. If there are an infinite number of such points, then there must exist on the line

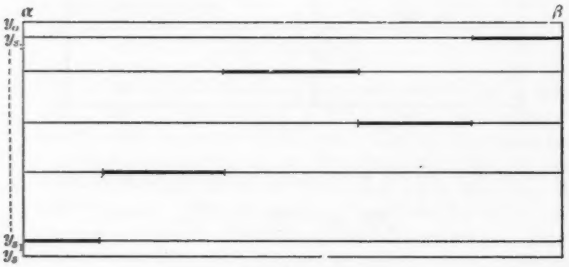

FIG. 3.

$y = y_0$ at least one limiting point of the corresponding values of x. This, however, can be avoided by the proper selection of the segments. For, let x_0 be such a limiting point. As we have seen, the value of $\Delta'(x) + \Delta(x)$ for $x = x_0$ is at least equal to l. Hence by the proper selection of a segment, we have for x_0 a neighborhood on $y = y_0$ equal to or greater than l which is entirely free from the end points mentioned above, and this is contrary to the supposition that they must of necessity become dense at some point.

It follows then that having selected an arbitrarily small positive number σ and a y_s at pleasure, we may choose a number $l > 0$ and lay off on the lines

$$y = y_{s_1}, y_{s_2}, \cdots y_{s_p},$$

$(y_{s_1}, y_{s_2}, \cdots y_{s_p})$ being taken between y_s and y_0, a finite number of segments fulfilling the requirements of the theorem. In case any segment extends beyond the given interval (α, β), we take only that portion of it which falls within (α, β).

The condition given by the theorem is also sufficient. To show this, let us suppose this condition fulfilled, and show that $f(x, y_0)$ is a continuous function for the interval $\alpha \leqq x \leqq \beta$. Let x_0 be any value of x within this interval and let y_s be any value chosen from the set (y) dense at y_0. If we assign to y_s any particular value, say y_{s_1}, then by hypothesis there exists a finite number of segments of lines lying between $y = y_{s_1}$ and $y = y_0$ fulfilling the condition of the theorem. Upon some one of these lines, say $y = y_{t_1}$, there exists for x_0 a neighborhood for which

$$|f(x, y_0) - f(x, y_{t_1})| < \sigma.$$

Again, if we put $y_s = y_{s_2}$, where y_{s_2} lies between y_{t_1} and y_0, there must exist a finite number of segments of lines lying between $y = y_{s_2}$ and $y = y_0$ likewise satisfying the condition of the theorem. Upon some one of these lines, say $y = y_{t_2}$, there exists for x_0 a neighborhood for which

$$|f(x, y_0) - f(x, y_{t_2})| < \sigma.$$

Continuing in the same manner, we obtain for x_0 an infinite succession of neighborhoods each point of which satisfies the inequality

$$|f(x, y_0) - f(x, y_{t_n})| < \sigma,$$

where $\lim_{n=\infty} y_{t_n} = y_0$. These successive neighborhoods may vary in extent with y_{t_n}, but from some point on they are all different from zero. The condition of theorem I is therefore satisfied for the point $x = x_0$ and consequently $f(x, y_0)$ is continuous at that point. But x_0 was any point of the given interval $\alpha \leqq x \leqq \beta$, and hence $f(x, y_0)$ is continuous throughout the interval. With this, the demonstration is completed.

§ 4. As we saw in §1, the representation of a function by means of an infinite series of continuous functions may be regarded as a special case of the problem considered in §§ 2, 3. Theorem I, giving the necessary and sufficient condition that the limiting function shall be continuous for the value $x = x_0$ becomes the following for the case of infinite series.

I′. *Given an infinite series*

$$u_1(x) + u_2(x) + \cdots + u_n(x) + \cdots$$

whose terms are continuous functions of x for the interval $\alpha \leqq x \leqq \beta$, and which converges for every value of x within the same interval. In order that the function $f(x)$ defined by this series be continuous at a point x_0 of the given interval, it is necessary and sufficient that for an arbitrarily small positive number σ and for every value of n greater than a sufficiently large number m there shall exist a neighborhood, which, however, may vary in extent with n, such that for every value of x within it we have

$$| f(x) - S_n(x) | \equiv | R_n(x) | < \sigma,$$

where $S_n(x)$ denotes the sum of the first n terms of the series and $R_n(x)$ is the remainder.

From theorem II we obtain the necessary and sufficient condition that $f(x)$ shall be continuous throughout the given interval $\alpha \leqq x \leqq \beta$. This condition may be stated for the infinite series as follows:

II′. *Given an infinite series of continuous functions which converges toward a limit $f(x)$ for each value of x within a definite interval $\alpha \leqq x \leqq \beta$. In order that $f(x)$ shall be continuous throughout this interval, it is necessary and sufficient that for an arbitrarily small positive number σ and for any integer m_1 there shall exist another integer $m_2 > m_1$ such that for some integer m lying between m_1 and m_2 we have*

$$| f(x) - S_m(x) | \equiv | R_m(x) | < \sigma, \quad \alpha \leqq x \leqq \beta,$$

where, however, m may change its value a finite number of times as x varies from α to β.

When we regard x and $1/n$ as the rectangular coördinates of a point in a plane, the above theorem may be stated more clearly perhaps as follows, where we make the same assumptions as

before concerning the nature of the terms of the infinite series and its convergence.

II''. *An arbitrarily small positive number σ and an integer m being chosen entirely at pleasure, then in order that $f(x)$, defined as above, shall be continuous throughout the interval $\alpha \leqq x \leqq \beta$, it is necessary and sufficient that there shall exist a finite number of segments of the lines*

$$n = m + p_1, \quad m + p_2, \cdots, \quad m + p_r$$

fulfilling the following conditions: The sum of the projections of these segments upon the x-axis shall fill the entire interval $\alpha \leqq x \leqq \beta$, and every point $(x, 1/n)$ of these segments shall satisfy the inequality

$$|f(x) - S_n(x)| \equiv |R_n(x)| < \sigma.$$

Theorems II', II'' introduce a kind of convergence of infinite series which Arzelà has called uniform convergence by segments (convergenza uniforme a tratti), although sub-uniform convergence seems a more appropriate name.* It differs from ordinary uniform convergence in that for uniform convergence each segment must fill the entire interval in question, and, moreover, such a segment must be present for every value of n greater than some given integer. As we have seen, neither of these conditions has to be fulfilled for the case of sub-uniform convergence. In fact, a series might converge sub-uniformly in a given interval and yet not be uniformly convergent in any sub-division of that interval, however small.

On the other hand, sub-uniform convergence differs from the simple uniform convergence (einfach gleichmässige Convergenz) introduced by Dini.† While Dini's simple uniform convergence is like sub-uniform convergence in that it does not require that the segments mentioned in the theorem shall be present for all values of n from a certain point on, it differs from sub-uniform convergence in requiring that these segments shall each fill the entire interval in question. From these considerations it follows that when we have uniform convergence of a series in the ordinary sense, or simple uniform convergence in the Dini sense, the series converges sub-uniformly in the Arzelà sense. The converse, however, is not true. Since sub-uniform convergence is the necessary and sufficient condition for the continuity of

* See also Moore: BULLETIN, vol. 7, March, 1901, p. 257.
† See: Grundlagen, etc., p. 137.

the limiting function, it follows from what has been said that any series which converges uniformly or simply uniformly defines a continuous function, provided the terms of the series are also continuous functions within the given closed interval.

The distinction between these various kinds of convergence is illustrated by the following examples.

1. Let $f(x)$ be defined by the series

$$1 + x + x^2 + x^3 + \cdots + x^n + \cdots, \qquad 0 \leqq x \leqq \tfrac{1}{2}.$$

We have then

$$S_n(x) = \frac{x^n - 1}{x - 1}, \quad f(x) = \frac{1}{1 - x}, \quad |R_n(x)| = \frac{x^n}{x - 1}.$$

If now we select σ arbitrarily small, we can determine a value of n, say n_1, such that for $n > n_1$ we have

$$|R_n(x)| < \sigma$$

for all values of x for which the series is defined. In other words, if we consider any of the points on the lines given by putting

$$n = n_1, \quad n_1 + 1, \quad n_1 + 2, \cdots$$

the above inequality holds for all values of x within the given interval. The series converges uniformly therefore within this interval. We have also simple uniform convergence, since the projection of each of these segments upon the x-axis fills the entire interval $0 \leqq x \leqq \frac{1}{2}$ and the corresponding values of n from a set dense at infinity. Moreover, the series converges sub-uniformly; for it is at once evident that the conditions of theorem II'' are satisfied.

2. Given the series whose terms are formed in accordance with the following law:

$$u_{2m-1} = \frac{x}{mx^2 + (1 - mx)^2},$$

$$u_{2m} = \frac{-x}{(m+1)x^2 + [1 - (m+1)x]^2},$$

$$(m = 1, 2, 3, \cdots; \ -\tfrac{1}{2} \leqq x \leqq \tfrac{1}{2}).$$

The series converges for all values of x within the given interval and gives

$$f(x) = \frac{x}{x^2 + 1 - x}.$$

When n is odd, say of the form $2m - 1$, we have

$$S_n(x) = \frac{x}{x^2 + (1-x)^2}, \quad R_n(x) = 0.$$

Hence for all odd values of n, however large, we have

$$|R_n(x)| < \sigma.$$

On the other hand, when n is even, say of the form $2m - 2$, we have

$$S_n(x) = \frac{x}{x^2+(1-x)^2} - \frac{x}{mx^2+(1-mx)^2}, \; |R_n(x)| = \frac{x}{mx^2+(1-mx)^2}$$

As n takes increasing even values, it is impossible to find any integer beyond which for all such values of n, and for all values of x within the given interval, the inequality

$$|R_n(x)| < \sigma$$

shall exist when we choose $\sigma < 1$. This follows from the fact that the values of $|R_n(x)|$ approach 1 along the line $x = 1/m$ as n approaches infinity. Hence for $\sigma < 1$, those segments which fill the entire interval $-\frac{1}{2} \leq x \leq \frac{1}{2}$, and for each point of which the above inequality is valid, can exist only for odd values of n. These values of n are, however, dense at infinity. Hence we have simple uniform convergence, but not ordinary uniform convergence. However, the conditions of theorem II'' are satisfied and the series converges sub-uniformly.

3. Given the series, the sum of whose first n terms is

$$S_n(x) = \frac{nx}{1 + n^2x^2}, \quad -1 \leq x \leq +1$$

This series converges for each value of x within this interval and gives

$$f(x) = \lim_{n=\infty} S_n(x) = 0.$$

For any finite value of n, $S_n(x)$ at one point on the line $x = 1/n$ is equal to $\frac{1}{2}$ and at one point on the line $x = -1/n$ is equal to $-\frac{1}{2}$. Hence, for $\sigma < \frac{1}{2}$, there are no segments filling the entire interval $(-1, +1)$ such that for each point of them we have

$$|R_n(x)| < \sigma.$$

It follows that the series does not converge uniformly in the given interval, nor do we have simple uniform convergence. However, the series does converge sub-uniformly; for we may, for any arbitrary σ, select the required segments as follows.

First of all, consider the series for values of x in the neighborhood of $x = 0$. For $x = 0$, we have

$$f(0) = \lim_{n=\infty} S_n(0) = 0.$$

Hence there exists a definite value of n, say $n = n_1$, such that

$$|f(0) - S_{n_1}(0)| < \frac{\sigma}{3}. \tag{1}$$

Because $S_n(x)$ is continuous in x, there exists upon the line $n = n_1$ an interval $(0 - \delta_{n_1}, 0 + \delta'_{n_1})$ such that for every value of x within it, we have

$$|S_{n_1}(0) - S_{n_1}(x)| < \frac{\sigma}{3}. \tag{2}$$

Moreover, since $f(x)$ equals zero for all values of x within the given interval $(-1, +1)$, we have for all values of x under consideration the inequality

$$|f(x) - f(0)| < \frac{\sigma}{3}. \tag{3}$$

By combining the inequalities (1), (2), (3), we obtain the following relation

$$|f(x) - S_1(x)| \equiv |R_{n_1}(x)| < \sigma, \tag{4}$$

which holds for all values of x within the interval $(0 - \delta_{n_1}, 0 + \delta'_{n_1})$ taken on the line $n = n_1$. Excluding this sub-interval from consideration, the given series converges uniformly through-

out the remaining portion of the interval $(-1, +1)$. Hence, for the same value of σ, we can find on some line $n = n_2$, $n_2 > n_1$, a segment filling the entire remaining part of the given interval which lies to the right of $x = \delta_{n_1}$, such that at each point of this segment the inequality

$$|R_{n_2}(x)| < \sigma$$

is valid. For the same reason, we can find on some line $n = n_3$, $n_3 > n_1$, a segment filling the remaining part of the given interval to the left of the point $x = -\delta_{n_1}$, such that for each point of this segment the above inequality holds.

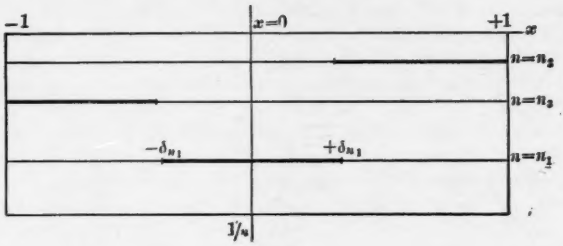

FIG. 4.

The three segments taken together satisfy the conditions of theorem II′, and hence the series converges sub-uniformly in the interval $-1 \leqq x \leqq 1$.

UNIVERSITY OF ILLINOIS.

GALOIS FIELD TABLES FOR $p^n \leqq 169$.

BY DR. W. H. BUSSEY.

(Read before the American Mathematical Society, September 7, 1905).

EVERY field of a finite number of marks may be represented as a Galois field of order $s = p^n$, where p^n is a power of a prime. The $GF[p^n]$ is defined uniquely by its order, and is therefore independent of the particular irreducible congruence used in its construction. In each of the following tables the $GF[p^n]$ is constructed by means of a primitive irreducible congruence which appears at the top of the table. The marks of each field are arranged in two tables. In each table each mark appears as a power of a primitive root i, and also as a polynomial in i of degree $k \leqq n-1$. The coefficients in this polynomial are integers reduced modulo p. The mark $Ai^k + Bi^{k-1} + \cdots + Di + E$, $A \neq 0$ is denoted by $AB\cdots DE$, a symbol consisting of its detached coefficients in order. Zero coefficients must not be omitted. This is the usual symbol for a positive integer in the notation of the number system whose base is p. In the first table the marks are arranged according to ascending powers of i. In the second table the marks are arranged so that the symbols $AB\cdots DE$ represent the positive integers in natural order. By means of these two tables it is possible to perform with ease the operations of addition, subtraction, multiplication and division, within the field.

For an exposition of the Galois field theory, see Dickson's Linear Groups, pages 1–54; Jordan's Traité des Substitutions, pages 14–18, pages 156–161; Serret's Algèbre supérieure. For other references on Galois fields and higher irreducible congruences, see the preface to Dickson's Linear Groups.

Example 1. Simplify $(i^7 + i^{13})(i^2 + 3i + 4)$, i being a primitive root of the $GF[7^2]$.

From the first table for $GF[7^2]$, $i^7 = 6i + 1$, $i^{13} = 3i + 3$, $i^2 = i + 4$.

Therefore $i^7 + i^{13} = 9i + 4 = 2i + 4 \pmod 7 = i^6$ (by second table).

Also $i^2 + 3i + 4 = 4i + 8 = 4i + 1 \pmod 7 = i^{22}$ (by second table).

Therefore $(i^7 + i^{13})(i^2 + 3i + 4) = i^6 \cdot i^{22} = i^{28} = 5i + 1$ (by first table).

Example 2. Simplify $\dfrac{i^{11}+3i^{79}}{2i^{26}-3i^{8}+2}$, i being a primitive root of the $GF[5^3]$.

From the first table for $GF[5^3]$, $i^{11}=3i^2+3i$, $i^{79}=3i^2+i+1$, $i^{26}=3i+1$, $-i^8=i^{70}=3i^2+4i+4$.

Therefore $i^{11}+3i^{79}=3i^2+3i+3(3i^2+i+1)=12i^2+6i+3=2i+i+3$ (mod. 5) $=i^{81}$ (by second table).

Also $2i^{26}-3i^8+2=2(3i+1)+3(3i^2+4i+4)+2=9i^2+18i+16=4i^2+3i+1$ (mod. 5) $=i^{18}$ (by second table).

Therefore

$$\frac{i^{11}+i^{79}}{2i^{26}-3i^{8}+2}=\frac{i^{81}}{i^{18}}=i^{63}=4i \text{ (by first table).}$$

Example 3. Simplify $\dfrac{i^{88}-10i^7+4}{i^{83}+2/i^{117}}$, i being a primitive root of the $GF[11^2]$.

From the first table for $GF[11^2]$, $i^{88}=6i+2$, $i^7=9i+2$, $i^{83}=2i+3$, $i^3=3i+3$.

Also $-10i^7=i^7$, because $-10\equiv 1$ modulo 11; and $2/i^{117}=2i^3$ because $i^{120}=1$.

Therefore $i^{88}-10i^7+4=(6i+2)+(9i+2)+4=15i+8=4i+8$ (modulo 11) $=i^{82}$ (by second table).

Also $i^{83}+2/i^{117}=(2i+3)+2(3i+3)=8i+9=i^{101}$ (by second table).

Therefore

$$\frac{i^{88}-10i^7+4}{i^{83}+2/i^{117}}=\frac{i^{82}}{i^{101}}=\frac{i^{82}\cdot i^{120}}{i^{101}}=\frac{i^{202}}{i^{101}}=i^{101}=8i+9 \text{ (by first table).}$$

$GF[2^3]$, $i^3\equiv i+1$, modulo 2. $i^\lambda=\alpha i^2+\beta i+\gamma$.

First Table.

λ	α	β	γ
1		1	0
2	1	0	0
3		1	1
4	1	1	0
5	1	1	1
6	1	0	1
7			1

Second Table.

λ	α	β	γ
7			1
1		1	0
3		1	1
2	1	0	0
6	1	0	1
4	1	1	0
5	1	1	1

$GF[3^2]$, $i^2 \equiv i+1$, modulo 3. $i^\lambda = \alpha i + \beta$.

FIRST TABLE.

λ	α	β	λ	α	β
1	1	0	5	2	0
2	1	1	6	2	2
3	2	1	7	1	2
4		2	8		1

SECOND TABLE.

λ	α	β	λ	α	β
8		1	7	1	2
4		2	5	2	0
1	1	0	3	2	1
2	1	1	6	2	2

$GF[2^4]$, $i^4 \equiv i+1$, modulo 2. $i^\lambda = \alpha i^3 + \beta i^2 + \gamma i + \delta$.

FIRST TABLE.

λ	α	β	γ	δ
1			1	0
2		1	0	0
3	1	0	0	0
4			1	1
5		1	1	0
6	1	1	0	0
7	1	0	1	1
8		1	0	1
9	1	0	1	0
10		1	1	1
11	1	1	1	0
12	1	1	1	1
13	1	1	0	1
14	1	0	0	1
15				1

SECOND TABLE.

λ	α	β	γ	δ
15				1
1			1	0
4			1	1
2		1	0	0
8		1	0	1
5		1	1	0
10		1	1	1
3	1	0	0	0
14	1	0	0	1
9	1	0	1	0
7	1	0	1	1
6	1	1	0	0
13	1	1	0	1
11	1	1	1	0
12	1	1	1	1

$GF[5^2]$, $i^2 \equiv 2i+2$, modulo 5. $i^\lambda = \alpha i + \beta$.

FIRST TABLE.

λ	α	β	λ	α	β
1	1	0	13	4	0
2	2	2	14	3	3
3	1	4	15	4	1
4	1	2	16	4	3
5	4	2	17	1	3
6		3	18		2
7	3	0	19	2	0
8	1	1	20	4	4
9	3	2	21	2	3
10	3	1	22	2	4
11	2	1	23	3	4
12		4	24		1

SECOND TABLE.

λ	α	β	λ	α	β
24		1	21	2	3
18		2	22	2	4
6		3	7	3	0
12		4	10	3	1
1	1	0	9	3	2
8	1	1	14	3	3
4	1	2	23	3	4
17	1	3	13	4	0
3	1	4	15	4	1
19	2	0	5	4	2
11	2	1	16	4	3
2	2	2	20	4	4

$GF[2^5]$, $i^5 \equiv i^3 + i^2 + i + 1$, modulo 2. $i^\lambda = \alpha i^4 + \beta i^3 + \gamma i^2 + \delta i + \epsilon$.

FIRST TABLE.

λ	α	β	γ	δ	ϵ
1				1	0
2			1	0	0
3		1	0	0	0
4	1	0	0	0	0
5		1	1	1	1
6	1	1	1	1	0
7	1	0	0	1	1
8		1	0	0	1
9	1	0	0	1	0
10		1	0	1	1
11	1	0	1	1	0
12				1	1
13			1	1	0
14		1	1	0	0
15	1	1	0	0	0
16	1	1	1	1	1
17	1	0	0	0	1
18		1	1	0	1
19	1	1	0	1	0
20	1	1	0	1	1
21	1	1	0	0	1
22	1	1	1	0	1
23	1	0	1	0	1
24			1	0	1
25		1	0	1	0
26	1	0	1	0	0
27			1	1	1
28		1	1	1	0
29	1	1	1	0	0
30	1	0	1	1	1
31					1

SECOND TABLE.

λ	α	β	γ	δ	ϵ
31					1
1				1	0
12				1	1
2			1	0	0
24			1	0	1
13			1	1	0
27			1	1	1
3		1	0	0	0
8		1	0	0	1
25		1	0	1	0
10		1	0	1	1
14		1	1	0	0
18		1	1	0	1
28		1	1	1	0
5		1	1	1	1
4	1	0	0	0	0
17	1	0	0	0	1
9	1	0	0	1	0
7	1	0	0	1	1
26	1	0	1	0	0
23	1	0	1	0	1
11	1	0	1	1	0
30	1	0	1	1	1
15	1	1	0	0	0
21	1	1	0	0	1
19	1	1	0	1	0
20	1	1	0	1	1
29	1	1	1	0	0
22	1	1	1	0	1
6	1	1	1	1	0
16	1	1	1	1	1

$GF[3^3]$, $i^3 \equiv i + 2$, modulo 3. $i^\lambda = \alpha i^2 + \beta i + \gamma$.

FIRST TABLE.

λ	α	β	γ	λ	α	β	γ	λ	α	β	γ
1		1	0	10	1	1	0	19	2	2	2
2	1	0	0	11	1	1	2	20	2	1	1
3		1	2	12	1	0	2	21	1	0	1
4	1	2	0	13			2	22		2	2
5	2	1	2	14		2	0	23	2	2	0
6	1	1	1	15	2	0	0	24	2	2	1
7	1	2	2	16		2	1	25	2	0	1
8	2	0	2	17	2	1	0	26			1
9		1	1	18	1	2	1				

$GF[3^3]$, $i^\lambda = \alpha i^2 + \beta i + \gamma$.

SECOND TABLE.

λ	α	β	γ	λ	α	β	γ	λ	α	β	γ
26			1	21	1	0	1	25	2	0	1
13			2	12	1	0	2	8	2	0	2
1		1	0	10	1	1	0	17	2	1	0
9		1	1	6	1	1	1	20	2	1	1
3		1	2	11	1	1	2	5	2	1	2
14		2	0	4	1	2	0	23	2	2	0
16		2	1	18	1	2	1	24	2	2	1
22		2	2	7	1	2	2	19	2	2	2
2	1	0	0	15	2	0	0				

$GF[7^2]$, $i^2 \equiv i + 4$, modulo 7. $i^\lambda = \alpha i + \beta$.

FIRST TABLE.

λ	α	β	λ	α	β
1	1	0	25	6	0
2	1	4	26	6	3
3	5	4	27	2	3
4	2	6	28	5	1
5	1	1	29	6	6
6	2	4	30	5	3
7	6	1	31	1	6
8		3	32		4
9	3	0	33	4	0
10	3	5	34	4	2
11	1	5	35	6	2
12	6	4	36	1	3
13	3	3	37	4	4
14	6	5	38	1	2
15	4	3	39	3	4
16		2	40		5
17	2	0	41	5	0
18	2	1	42	5	6
19	3	1	43	4	6
20	4	5	44	3	2
21	2	2	45	5	5
22	4	1	46	3	6
23	5	2	47	2	5
24		6	48		1

SECOND TABLE.

λ	α	β	λ	α	β
48		1	39	3	4
16		2	10	3	5
8		3	46	3	6
32		4	33	4	0
40		5	22	4	1
24		6	34	4	2
1	1	0	15	4	3
5	1	1	37	4	4
38	1	2	20	4	5
36	1	3	43	4	6
2	1	4	41	5	0
11	1	5	28	5	1
31	1	6	23	5	2
17	2	0	30	5	3
18	2	1	3	5	4
21	2	2	45	5	5
27	2	3	42	5	6
6	2	4	25	6	0
47	2	5	7	6	1
4	2	6	35	6	2
9	3	0	26	6	3
19	3	1	12	6	4
44	3	2	14	6	5
13	3	3	29	6	6

$GF[2^6]$, $i^6 \equiv i+1$, modulo 2. $i^\lambda = \alpha i^5 + \beta i^4 + \gamma i^3 + \delta i^2 + \epsilon i + \zeta$.

FIRST TABLE.

λ	α	β	γ	δ	ϵ	ζ	λ	α	β	γ	δ	ϵ	ζ
1					1	0	33		1	0	0	1	0
2				1	0	0	34	1	0	0	1	0	0
3			1	0	0	0	35			1	0	1	1
4		1	0	0	0	0	36		1	0	1	1	0
5	1	0	0	0	0	0	37	1	0	1	1	0	0
6					1	1	38		1	1	0	1	1
7				1	1	0	39	1	1	0	1	1	0
8			1	1	0	0	40	1	0	1	1	1	1
9		1	1	0	0	0	41		1	1	1	0	1
10	1	1	0	0	0	0	42	1	1	1	0	1	0
11	1	0	0	0	1	1	43	1	1	0	1	1	1
12				1	0	1	44	1	0	1	1	0	1
13			1	0	1	0	45		1	1	0	0	1
14		1	0	1	0	0	46	1	1	0	0	1	0
15	1	0	1	0	0	0	47	1	0	0	1	1	1
16		1	0	0	1	1	48			1	1	0	1
17	1	0	0	1	1	0	49		1	1	0	1	0
18			1	1	1	1	50	1	1	0	1	0	0
19		1	1	1	1	0	51	1	0	1	0	1	1
20	1	1	1	1	0	0	52		1	0	1	0	1
21	1	1	1	0	1	1	53	1	0	1	0	1	0
22	1	1	0	1	0	1	54		1	0	1	1	1
23	1	0	1	0	0	1	55	1	0	1	1	1	0
24		1	0	0	0	1	56		1	1	1	1	1
25	1	0	0	0	1	0	57	1	1	1	1	1	0
26				1	1	1	58	1	1	1	1	1	1
27			1	1	1	0	59	1	1	1	1	0	1
28		1	1	1	0	0	60	1	1	1	0	0	1
29	1	1	1	0	0	0	61	1	1	0	0	0	1
30	1	1	0	0	1	1	62	1	0	0	0	0	1
31	1	0	0	1	0	1	63						1
32			1	0	0	1							

SECOND TABLE.

λ	α	β	γ	δ	ϵ	ζ	λ	α	β	γ	δ	ϵ	ζ
63						1	27			1	1	1	0
1					1	0	18			1	1	1	1
6					1	1	4		1	0	0	0	0
2				1	0	0	24		1	0	0	0	1
12				1	0	1	33		1	0	0	1	0
7				1	1	0	16		1	0	0	1	1
26				1	1	1	14		1	0	1	0	0
3			1	0	0	0	52		1	0	1	0	1
32			1	0	0	1	36		1	0	1	1	0
13			1	0	1	0	54		1	0	1	1	1
35			1	0	1	1	9		1	1	0	0	0
8			1	1	0	0	45		1	1	0	0	1
48			1	1	0	1	49		1	1	0	1	0

$GF[2^6]$, $i^6 \equiv i + 1$, modulo 2. $i^\lambda = \alpha i^5 + \beta i^4 + \gamma i^3 + \delta i^2 + \epsilon i + \zeta$.

SECOND TABLE.—*Continued.*

λ	α	β	γ	δ	ϵ	ζ	λ	α	β	γ	δ	ϵ	ζ
38		1	1	0	1	1	55	1	0	1	1	1	0
28		1	1	1	0	0	40	1	0	1	1	1	1
41		1	1	1	0	1	10	1	1	0	0	0	0
19		1	1	1	1	0	61	1	1	0	0	0	1
56		1	1	1	1	1	46	1	1	0	0	1	0
5	1	0	0	0	0	0	30	1	1	0	0	1	1
62	1	0	0	0	0	1	50	1	1	0	1	0	0
25	1	0	0	0	1	0	22	1	1	0	1	0	1
11	1	0	0	0	1	1	39	1	1	0	1	1	0
34	1	0	0	1	0	0	43	1	1	0	1	1	1
31	1	0	0	1	0	1	29	1	1	1	0	0	0
17	1	0	0	1	1	0	60	1	1	1	0	0	1
47	1	0	0	1	1	1	42	1	1	1	0	1	0
15	1	0	1	0	0	0	21	1	1	1	0	1	1
23	1	0	1	0	0	1	20	1	1	1	1	0	0
53	1	0	1	0	1	0	59	1	1	1	1	0	1
51	1	0	1	0	1	1	57	1	1	1	1	1	0
37	1	0	1	1	0	0	58	1	1	1	1	1	1
44	1	0	1	1	0	1							

$GF[3^4]$, $i^4 \equiv 2i^3 + 2i^2 + i + 1$, modulo 3. $i^\lambda = \alpha i^3 + \beta i^2 + \gamma i + \delta$.

FIRST TABLE.

λ	α	β	γ	δ	λ	α	β	γ	δ
1			1	0	24		2	2	2
2		1	0	0	25	2	2	2	0
3	1	0	0	0	26			2	2
4	2	2	1	1	27		2	2	0
5		2	0	2	28	2	2	0	0
6	2	0	2	0	29		1	2	2
7	1	0	2	2	30	1	2	2	0
8	2	1	0	1	31	1	1	1	1
9	2	1	0	2	32			2	1
10	2	1	1	2	33		2	1	0
11	2	2	1	2	34	2	1	0	0
12		2	1	2	35	2	1	2	2
13	2	1	2	0	36	2	0	1	2
14	2	0	2	2	37	1	2	1	2
15	1	0	1	2	38	1	0	0	1
16	2	0	0	1	39	2	2	2	1
17	1	1	0	2	40				2
18		2	0	1	41			2	0
19	2	0	1	0	42		2	0	0
20	1	2	2	2	43	2	0	0	0
21	1	1	0	1	44	1	1	2	2
22		2	2	1	45		1	0	1
23	2	2	1	0	46	1	0	1	0

$GF[3^4]$, $i^4 \equiv 2i^3 + 2i^2 + i + 1$, modulo 3. $i^\lambda = \alpha i^3 + \beta i^2 + \gamma i + \delta$.

FIRST TABLE.—*Continued.*

λ	α	β	γ	δ	λ	α	β	γ	δ
47	2	0	1	1	64		1	1	1
48	1	2	0	2	65	1	1	1	0
49	1	2	0	1	66			1	1
50	1	2	2	1	67		1	1	0
51	1	1	2	1	68	1	1	0	0
52		1	2	1	69		2	1	1
53	1	2	1	0	70	2	1	1	0
54	1	0	1	1	71	2	2	2	2
55	2	0	2	1	72			1	2
56	1	0	0	2	73		1	2	0
57	2	2	0	1	74	1	2	0	0
58		1	0	2	75	1	2	1	1
59	1	0	2	0	76	1	0	2	1
60	2	1	1	1	77	2	1	2	1
61	2	2	0	2	78	2	0	0	2
62		1	1	2	79	1	1	1	2
63	1	1	2	0	80				1

SECOND TABLE.

λ	α	β	γ	δ	λ	α	β	γ	δ
80				1	3	1	0	0	0
40				2	38	1	0	0	1
1			1	0	56	1	0	0	2
66			1	1	46	1	0	1	0
72			1	2	54	1	0	1	1
41			2	0	15	1	0	1	2
32			2	1	59	1	0	2	0
26			2	2	76	1	0	2	1
2		1	0	0	7	1	0	2	2
45		1	0	1	68	1	1	0	0
58		1	0	2	21	1	1	0	1
67		1	1	0	17	1	1	0	2
64		1	1	1	65	1	1	1	0
62		1	1	2	31	1	1	1	1
73		1	2	0	79	1	1	1	2
52		1	2	1	63	1	1	2	0
29		1	2	2	51	1	1	2	1
42		2	0	0	44	1	1	2	2
18		2	0	1	74	1	2	0	0
5		2	0	2	49	1	2	0	1
33		2	1	0	48	1	2	0	2
69		2	1	1	53	1	2	1	0
12		2	1	2	75	1	2	1	1
27		2	2	0	37	1	2	1	2
22		2	2	1	30	1	2	2	0
24		2	2	2					

$GF[3^4]$, $i^4 \equiv 2i^3 + 2i^2 + i + 1$, modulo 3.

SECOND TABLE.—*Continued.*

λ	α	β	γ	δ	λ	α	β	γ	δ
50	1	2	2	1	60	2	1	1	1
20	1	2	2	2	10	2	1	1	2
43	2	0	0	0	13	2	1	2	0
16	2	0	0	1	77	2	1	2	1
78	2	0	0	2	35	2	1	2	2
19	2	0	1	0	28	2	2	0	0
47	2	0	1	1	57	2	2	0	1
36	2	0	1	2	61	2	2	0	2
6	2	0	2	0	23	2	2	1	0
55	2	0	2	1	4	2	2	1	1
14	2	0	2	2	11	2	2	1	2
34	2	1	0	0	25	2	2	2	0
8	2	1	0	1	39	2	2	2	1
9	2	1	0	2	71	2	2	2	2
70	2	1	1	0					

$GF[11^2]$, $i^2 \equiv 4i + 9$, modulo 11. $i^\lambda = \alpha i + \beta$.

FIRST TABLE.

λ	α	β	λ	α	β	λ	α	β	λ	α	β
1	1	0	31	3	8	61	10	0	91	8	3
2	4	9	32	9	5	62	7	2	92	2	6
3	3	3	33	8	4	63	8	8	93	3	7
4	4	5	34	3	6	64	7	6	94	8	5
5	10	3	35	7	5	65	1	8	95	4	6
6	10	2	36		8	66	1	9	96		3
7	9	2	37	8	0	67	2	9	97	3	0
8	5	4	38	10	6	68	6	7	98	1	5
9	2	1	39	2	2	69	9	10	99	9	9
10	9	7	40	10	7	70	2	4	100	1	4
11	10	4	41	3	2	71	1	7	101	8	9
12		2	42	3	5	72		9	102	8	6
13	2	0	43	6	5	73	9	0	103	5	6
14	8	7	44	7	10	74	3	4	104	4	1
15	6	6	45	5	8	75	5	5	105	6	3
16	8	10	46	6	1	76	3	1	106	5	10
17	9	6	47	3	10	77	2	5	107	8	1
18	9	4	48		5	78	2	7	108		6
19	7	4	49	5	0	79	4	7	109	6	0
20	10	8	50	9	1	80	1	3	110	2	10
21	4	2	51	4	4	81	7	9	111	7	7
22	7	3	52	9	3	82	4	8	112	2	8
23	9	8	53	6	4	83	2	3	113	5	7
24		4	54	6	10	84		7	114	5	1
25	4	0	55	1	10	85	7	0	115	10	1
26	5	3	56	3	9	86	6	8	116	8	2
27	1	1	57	10	5	87	10	10	117	1	6
28	5	9	58	1	2	88	6	2	118	10	9
29	7	1	59	6	9	89	4	10	119	5	2
30	7	8	60		10	90	4	3	120		1

$GF[11^2]$, $i^2 = 4i + 9$, modulo 11. $i^\lambda = \alpha i + \beta$.

SECOND TABLE.

λ	α	β	λ	α	β	λ	α	β	λ	α	β
120		1	67	2	9	103	5	6	91	8	3
12		2	110	2	10	113	5	7	33	8	4
96		3	97	3	0	45	5	8	94	8	5
24		4	76	3	1	28	5	9	102	8	6
48		5	41	3	2	106	5	10	14	8	7
108		6	3	3	3	109	6	0	63	8	8
84		7	74	3	4	46	6	1	101	8	9
36		8	42	3	5	88	6	2	16	8	10
72		9	34	3	6	105	6	3	73	9	0
60		10	93	3	7	53	6	4	50	9	1
1	1	0	31	3	8	43	6	5	7	9	2
27	1	1	56	3	9	15	6	6	52	9	3
58	1	2	47	3	10	68	6	7	18	9	4
80	1	3	25	4	0	86	6	8	32	9	5
100	1	4	104	4	1	59	6	9	17	9	6
98	1	5	21	4	2	54	6	10	10	9	7
117	1	6	90	4	3	85	7	0	23	9	8
71	1	7	51	4	4	29	7	1	99	9	9
65	1	8	4	4	5	62	7	2	69	9	10
66	1	9	95	4	6	22	7	3	61	10	0
55	1	10	79	4	7	19	7	4	115	10	1
13	2	0	82	4	8	35	7	5	6	10	2
9	2	1	2	4	9	64	7	6	5	10	3
39	2	2	89	4	10	111	7	7	11	10	4
83	2	3	49	5	0	30	7	8	57	10	5
70	2	4	114	5	1	81	7	9	38	10	6
77	2	5	119	5	2	44	7	10	40	10	7
92	2	6	26	5	3	37	8	0	20	10	8
78	2	7	8	5	4	107	8	1	118	10	9
112	2	8	75	5	5	116	8	2	87	10	10

$GF[5^3]$, $i^3 \equiv 2i + 3$, modulo 5. $i^\lambda = \alpha i^2 + \beta i + \gamma$.

FIRST TABLE.

λ	α	β	γ	λ	α	β	γ	λ	α	β	γ
1		1	0	13	1	0	4	25	2	0	4
2	1	0	0	14		1	3	26		3	1
3		2	3	15	1	3	0	27	3	1	0
4	2	3	0	16	3	2	3	28	1	1	4
5	3	4	1	17	2	4	4	29	1	1	3
6	4	2	4	18	4	3	1	30	1	0	3
7	2	2	2	19	3	4	2	31			3
8	2	1	1	20	4	3	4	32		3	0
9	1	0	1	21	3	2	2	33	3	0	0
10		3	3	22	2	3	4	34		1	4
11	3	3	0	23	3	3	1	35	1	4	0
12	3	1	4	24	3	2	4	36	4	2	3

$GF[5^3]$, $i^3 \equiv 2i + 3$, modulo 5. $i^\lambda = \alpha i^2 + \beta i + \gamma$.

FIRST TABLE.—*Continued.*

λ	α	β	γ	λ	α	β	γ	λ	α	β	γ
37	2	1	2	67	2	1	4	97	4	1	0
38	1	1	1	68	1	3	1	98	1	3	2
39	1	3	3	69	3	3	3	99	3	4	3
40	3	0	3	70	3	4	4	100	4	4	4
41		4	4	71	4	0	4	101	4	2	2
42	4	4	0	72		2	2	102	2	0	2
43	4	3	2	73	2	2	0	103		1	1
44	3	0	2	74	2	4	1	104	1	1	0
45		3	4	75	4	0	1	105	1	2	3
46	3	4	0	76		4	2	106	2	0	3
47	4	1	4	77	4	2	0	107		2	1
48	1	2	2	78	2	3	2	108	2	1	0
49	2	4	3	79	3	1	1	109	1	4	1
50	4	2	1	80	1	2	4	110	4	3	3
51	2	4	2	81	2	1	3	111	3	1	2
52	4	1	1	82	1	2	1	112	1	3	4
53	1	4	2	83	2	3	3	113	3	1	3
54	4	4	3	84	3	2	1	114	1	4	4
55	4	1	2	85	2	2	4	115	4	1	3
56	1	0	2	86	2	3	1	116	1	1	2
57		4	3	87	3	0	1	117	1	4	3
58	4	3	0	88		2	4	118	4	0	3
59	3	3	2	89	2	4	0	119		1	2
60	3	3	4	90	4	4	1	120	1	2	0
61	3	0	4	91	4	4	2	121	2	2	3
62			4	92	4	0	2	122	2	2	1
63		4	0	93			2	123	2	0	1
64	4	0	0	94		2	0	124			1
65		3	2	95	2	0	0				
66	3	2	0	96		4	1				

SECOND TABLE.

λ	α	β	γ	λ	α	β	γ	λ	α	β	γ
124			1	45		3	4	48	1	2	2
93			2	63		4	0	105	1	2	3
31			3	96		4	1	80	1	2	4
62			4	76		4	2	15	1	3	0
1		1	0	57		4	3	68	1	3	1
103		1	1	41		4	4	98	1	3	2
119		1	2	2	1	0	0	39	1	3	3
14		1	3	9	1	0	1	112	1	3	4
34		1	4	56	1	0	2	35	1	4	0
94		2	0	30	1	0	3	109	1	4	1
107		2	1	13	1	0	4	53	1	4	2
72		2	2	104	1	1	0	117	1	4	3
3		2	3	38	1	1	1	114	1	4	4
88		2	4	116	1	1	2	95	2	0	0
32		3	0	29	1	1	3	123	2	0	1
26		3	1	28	1	1	4	102	2	0	2
65		3	2	120	1	2	0	106	2	0	3
10		3	3	82	1	2	1	25	2	0	4

$GF[3^3]$, $i^3 \equiv 2i + 3$, modulo 5. $i^\lambda = \alpha i^2 + \beta i + \gamma$.

SECOND TABLE.—*Continued.*

λ	α	β	γ	λ	α	β	γ	λ	α	β	γ
108	2	1	0	61	3	0	4	118	4	0	3
8	2	1	1	27	3	1	0	71	4	0	4
37	2	1	2	79	3	1	1	97	4	1	0
81	2	1	3	111	3	1	2	52	4	1	1
67	2	1	4	113	3	1	3	55	4	1	2
73	2	2	0	12	3	1	4	115	4	1	3
122	2	2	1	66	3	2	0	47	4	1	4
7	2	2	2	84	3	2	1	77	4	2	0
121	2	2	3	21	3	2	2	50	4	2	1
85	2	2	4	16	3	2	3	101	4	2	2
4	2	3	0	24	3	2	4	36	4	2	3
86	2	3	1	11	3	3	0	6	4	2	4
78	2	3	2	23	3	3	1	58	4	3	0
83	2	3	3	59	3	3	2	18	4	3	1
22	2	3	4	69	3	3	3	43	4	3	2
89	2	4	0	60	3	3	4	110	4	3	3
74	2	4	1	46	3	4	0	20	4	3	4
51	2	4	2	5	3	4	1	42	4	4	0
49	2	4	3	19	3	4	2	90	4	4	1
17	2	4	4	99	3	4	3	91	4	4	2
33	3	0	0	70	3	4	4	54	4	4	3
87	3	0	1	64	4	0	0	100	4	4	4
44	3	0	2	75	4	0	1				
40	3	0	3	92	4	0	2				

$GF[2^7]$, $i^7 \equiv i + 1$, modulo 2. $i^{-\lambda} = \alpha i^6 + \beta i^5 + \gamma i^4 + \delta i^3 + \varepsilon i^2 + \zeta i + \eta$.

FIRST TABLE.

λ	α	β	γ	δ	ε	ζ	η	λ	α	β	γ	δ	ε	ζ	η
1						1	0	24	1	1	1	1	0	0	0
2				1	0	0		25	1	1	1	0	0	1	1
3				0	0	0		26	1	1	0	0	1	0	1
4			1	0	0	0		27	1	0	0	1	0	0	1
5		1	0	0	0	0		28			1	0	0	0	1
6	1	0	0	0	0	0		29		1	0	0	0	1	0
7						1	1	30	1	0	0	0	1	0	0
8				1	1	0		31				1	0	1	1
9				1	1	0		32			1	0	1	1	0
10			1	1	0	0		33		1	0	1	1	0	0
11		1	1	0	0	0		34	1	0	1	1	0	0	0
12	1	1	0	0	0	0		35		1	1	0	0	1	1
13	1	0	0	0	0	1	1	36	1	1	0	0	1	1	0
14					1	0	1	37	1	0	0	1	1	1	1
15				1	0	1	0	38			1	1	1	0	1
16			1	0	1	0	0	39		1	1	1	0	1	0
17		1	0	1	0	0	0	40	1	1	1	0	1	0	0
18	1	0	1	0	0	0	0	41	1	1	0	1	0	1	1
19		1	0	0	0	1	1	42	1	0	1	0	1	0	1
20	1	0	0	0	1	1	0	43		1	0	1	0	0	1
21				1	1	1	1	44	1	0	1	0	0	1	0
22			1	1	1	1	0	45		1	0	0	1	1	1
23		1	1	1	1	0	0	46	1	0	0	1	1	1	0

$GF[2^7]$, $\bar{i} \equiv i+1$, modulo 2. $i^{-\lambda} = \alpha i^6 + \beta i^5 + \gamma i^4 + \delta i^3 + \epsilon i^2 + \zeta i + \eta$.

FIRST TABLE.—*Continued.*

λ	α	β	γ	δ	ε	ζ	η	λ	α	β	γ	δ	ε	ζ	η
47			1	1	1	1	1	88	1	1	0	0	0	1	0
48		1	1	1	1	1	0	89	1	0	0	0	1	1	1
49	1	1	1	1	1	0	0	90				1	1	0	1
50	1	1	1	1	0	1	1	91			1	1	0	1	0
51	1	1	1	0	1	0	1	92		1	1	0	1	0	0
52	1	1	0	1	0	0	1	93	1	1	0	1	0	0	0
53	1	0	1	0	0	0	1	94	1	0	1	0	0	1	1
54		1	0	0	0	0	1	95		1	0	0	1	0	1
55	1	0	0	0	0	1	0	96	1	0	0	1	0	1	0
56					1	1	1	97			1	0	1	1	1
57				1	1	1	0	98		1	0	1	1	1	0
58			1	1	1	0	0	99	1	0	1	1	1	0	0
59		1	1	1	0	0	0	100		1	1	1	0	1	1
60	1	1	1	0	0	0	0	101	1	1	1	0	1	1	0
61	1	1	0	0	0	1	1	102	1	1	0	1	1	1	1
62	1	0	0	0	1	0	1	103	1	0	1	1	1	0	1
63				1	0	0	1	104		1	1	1	0	0	1
64			1	0	0	1	0	105	1	1	1	0	0	1	0
65		1	0	0	1	0	0	106	1	1	0	0	1	1	1
66	1	0	0	1	0	0	0	107	1	0	0	1	1	0	1
67			1	0	0	1	1	108			1	1	0	0	1
68		1	0	0	1	1	0	109		1	1	0	0	1	0
69	1	0	0	1	1	0	0	110	1	1	0	0	1	0	0
70			1	1	0	1	1	111	1	0	0	1	0	1	1
71		1	1	0	1	1	0	112			1	0	1	0	1
72	1	1	0	1	1	0	0	113		1	0	1	0	1	0
73	1	0	1	1	0	1	1	114	1	0	1	0	1	0	0
74		1	1	0	1	0	1	115		1	0	1	0	1	1
75	1	1	0	1	0	1	0	116	1	0	1	0	1	1	0
76	1	0	1	0	1	1	1	117		1	0	1	1	1	1
77		1	0	1	1	0	1	118	1	0	1	1	1	1	0
78	1	0	1	1	0	1	0	119		1	1	1	1	1	1
79		1	1	0	1	1	1	120	1	1	1	1	1	1	0
80	1	1	0	1	1	1	0	121	1	1	1	1	1	1	1
81	1	0	1	1	1	1	1	122	1	1	1	1	1	0	1
82		1	1	1	1	0	1	123	1	1	1	1	0	0	1
83	1	1	1	1	0	1	0	124	1	1	1	0	0	0	1
84	1	1	1	0	1	1	1	125	1	1	0	0	0	0	1
85	1	1	0	1	1	0	1	126	1	0	0	0	0	0	1
86	1	0	1	1	0	0	1	127							1
87		1	1	0	0	0	1								

$GF[2^7]$, $i^7 \equiv i+1$, modulo 2. $i^\nu = \alpha i^6 + \beta i^5 + \gamma i^4 + \delta i^3 + \epsilon i^2 + \zeta i + \eta$.

SECOND TABLE.

λ	α	β	γ	δ	ϵ	ζ	η	λ	α	β	γ	δ	ϵ	ζ	η
127							1	87		1	1	0	0	0	1
1						1	0	109		1	1	0	0	1	0
7						1	1	35		1	1	0	0	1	1
2					1	0	0	92		1	1	0	1	0	0
14					1	0	1	74		1	1	0	1	0	1
8					1	1	0	71		1	1	0	1	1	0
56					1	1	1	79		1	1	0	1	1	1
3				1	0	0	0	59		1	1	1	0	0	0
63				1	0	0	1	104		1	1	1	0	0	1
15				1	0	1	0	39		1	1	1	0	1	0
31				1	0	1	1	100		1	1	1	0	1	1
9				1	1	0	0	23		1	1	1	1	0	0
90				1	1	0	1	82		1	1	1	1	0	1
57				1	1	1	0	48		1	1	1	1	1	0
21				1	1	1	1	119		1	1	1	1	1	1
4			1	0	0	0	0	6	1	0	0	0	0	0	0
28			1	0	0	0	1	126	1	0	0	0	0	0	1
64			1	0	0	1	0	55	1	0	0	0	0	1	0
67			1	0	0	1	1	13	1	0	0	0	0	1	1
16			1	0	1	0	0	30	1	0	0	0	1	0	0
112			1	0	1	0	1	62	1	0	0	0	1	0	1
32			1	0	1	1	0	20	1	0	0	0	1	1	0
97			1	0	1	1	1	89	1	0	0	0	1	1	1
10			1	1	0	0	0	66	1	0	0	1	0	0	0
108			1	1	0	0	1	27	1	0	0	1	0	0	1
91			1	1	0	1	0	96	1	0	0	1	0	1	0
70			1	1	0	1	1	111	1	0	0	1	0	1	1
58			1	1	1	0	0	69	1	0	0	1	1	0	0
38			1	1	1	0	1	107	1	0	0	1	1	0	1
22			1	1	1	1	0	46	1	0	0	1	1	1	0
47			1	1	1	1	1	37	1	0	0	1	1	1	1
5		1	0	0	0	0	0	18	1	0	1	0	0	0	0
54		1	0	0	0	0	1	53	1	0	1	0	0	0	1
29		1	0	0	0	1	0	44	1	0	1	0	0	1	0
19		1	0	0	0	1	1	94	1	0	1	0	0	1	1
65		1	0	0	1	0	0	114	1	0	1	0	1	0	0
95		1	0	0	1	0	1	42	1	0	1	0	1	0	1
68		1	0	0	1	1	0	116	1	0	1	0	1	1	0
45		1	0	0	1	1	1	76	1	0	1	0	1	1	1
17		1	0	1	0	0	0	34	1	0	1	1	0	0	0
43		1	0	1	0	0	1	86	1	0	1	1	0	0	1
113		1	0	1	0	1	0	78	1	0	1	1	0	1	0
115		1	0	1	0	1	1	73	1	0	1	1	0	1	1
33		1	0	1	1	0	0	99	1	0	1	1	1	0	0
77		1	0	1	1	0	1	103	1	0	1	1	1	0	1
98		1	0	1	1	1	0	118	1	0	1	1	1	1	0
117		1	0	1	1	1	1	81	1	0	1	1	1	1	1
11		1	1	0	0	0	0	12	1	1	0	0	0	0	0

$GF[2^7]$, $i^7 \equiv i+1$, modulo 2. $i^\lambda = \alpha i^6 + \beta i^5 + \gamma i^4 + \delta i^3 + \epsilon i^2 + \zeta i + \eta$.

SECOND TABLE.—*Continued.*

λ	α	β	γ	δ	ε	ζ	η	λ	α	β	γ	δ	ε	ζ	η
125	1	1	0	0	0	0	1	124	1	1	1	0	0	0	1
88	1	1	0	0	0	1	0	105	1	1	1	0	0	1	0
61	1	1	0	0	0	1	1	25	1	1	1	0	0	1	1
110	1	1	0	0	1	0	0	40	1	1	1	0	1	0	0
26	1	1	0	0	1	0	1	51	1	1	1	0	1	0	1
36	1	1	0	0	1	1	0	101	1	1	1	0	1	1	0
106	1	1	0	0	1	1	1	84	1	1	1	0	1	1	1
93	1	1	0	1	0	0	0	24	1	1	1	1	0	0	0
52	1	1	0	1	0	0	1	123	1	1	1	1	0	0	1
75	1	1	0	1	0	1	0	83	1	1	1	1	0	1	0
41	1	1	0	1	0	1	1	50	1	1	1	1	0	1	1
72	1	1	0	1	1	0	0	49	1	1	1	1	1	0	0
85	1	1	0	1	1	0	1	122	1	1	1	1	1	0	1
80	1	1	0	1	1	1	0	120	1	1	1	1	1	1	0
102	1	1	0	1	1	1	1	121	1	1	1	1	1	1	1
60	1	1	1	0	0	0	0								

$GF[13^2]$, $i^2 \equiv i+11$, modulo 13. $i^\lambda = \alpha i + \beta$.

FIRST TABLE.

λ	α	β	λ	α	β	λ	α	β	λ	α	β
1	1	0	25	7	5	49	4	11	73	7	1
2	1	11	26	12	12	50	2	5	74	8	12
3	12	11	27	11	2	51	7	9	75	7	10
4	10	2	28		4	52	3	12	76	4	12
5	12	6	29	4	0	53	2	7	77	3	5
6	5	2	30	4	5	54	9	9	78	8	7
7	7	3	31	9	5	55	5	8	79	2	10
8	10	12	32	1	8	56		3	80	12	9
9	9	6	33	9	11	57	3	0	81	8	2
10	2	8	34	7	8	58	3	7	82	10	10
11	10	9	35	2	12	59	10	7	83	7	6
12	6	6	36	1	9	60	4	6	84		12
13	12	1	37	10	11	61	10	5	85	12	0
14		2	38	8	6	62	2	6	86	12	2
15	2	0	39	1	10	63	8	9	87	1	2
16	2	9	40	11	11	64	4	10	88	3	11
17	11	9	41	9	4	65	1	5	89	1	7
18	7	4	42		8	66	6	11	90	8	11
19	11	12	43	8	0	67	4	1	91	6	10
20	10	4	44	8	10	68	5	5	92	3	1
21	1	6	45	5	10	69	10	3	93	4	7
22	7	11	46	2	3	70		6	94	11	5
23	5	12	47	5	9	71	6	0	95	3	4
24	4	3	48	1	3	72	6	1	96	7	7

$GF[13^2]$, $i^2 \equiv i + 11$, modulo 13. $i^\lambda = \alpha i + \beta$.

FIRST TABLE.—*Continued.*

λ	α	β	λ	α	β	λ	α	β	λ	α	β
97	1	12	115	4	8	133	9	2	151	9	12
98		11	116	12	5	134	11	8	152	8	8
99	11	0	117	4	2	135	6	4	153	3	10
100	11	4	118	6	5	136	10	1	154		7
101	2	4	119	11	1	137	11	6	155	7	0
102	6	9	120	12	4	138	4	4	156	7	12
103	2	1	121	3	2	139	8	5	157	6	12
104	3	9	122	5	7	140		10	158	5	1
105	12	7	123	12	3	141	10	0	159	6	3
106	6	2	124	2	2	142	10	6	160	9	1
107	8	1	125	4	9	143	3	6	161	10	8
108	9	10	126		5	144	9	7	162	5	6
109	6	8	127	5	0	145	3	8	163	11	3
110	1	1	128	5	3	146	11	7	164	1	4
111	2	11	129	8	3	147	5	4	165	5	11
112		9	130	11	10	148	9	3	166	3	3
113	9	0	131	8	4	149	12	8	167	6	7
114	9	8	132	12	10	150	7	2	168		1

SECOND TABLE.

λ	α	β	λ	α	β	λ	α	β	⁻λ⁻	α	β
168		1	15	2	0	52	3	12	165	5	11
14		2	103	2	1	29	4	0	23	5	12
56		3	124	2	2	67	4	1	71	6	0
28		4	46	2	3	117	4	2	72	6	1
126		5	101	2	4	24	4	3	106	6	2
70		6	50	2	5	138	4	4	159	6	3
154		7	62	2	6	30	4	5	135	6	4
42		8	53	2	7	60	4	6	118	6	5
112		9	10	2	8	93	4	7	12	6	6
140		10	16	2	9	115	4	8	167	6	7
98		11	79	2	10	125	4	9	109	6	8
84		12	111	2	11	64	4	10	102	6	9
1	1	0	35	2	12	49	4	11	91	6	10
110	1	1	57	3	0	76	4	12	66	6	11
87	1	2	92	3	1	127	5	0	157	6	12
48	1	3	121	3	2	158	5	1	155	7	0
164	1	4	166	3	3	6	5	2	73	7	1
65	1	5	95	3	4	128	5	3	150	7	2
21	1	6	77	3	5	147	5	4	7	7	3
89	1	7	143	3	6	68	5	5	18	7	4
32	1	8	58	3	7	162	5	6	25	7	5
36	1	9	145	3	8	122	5	7	83	7	6
39	1	10	104	3	9	55	5	8	96	7	7
2	1	11	153	3	10	47	5	9	34	7	8
97	1	12	88	3	11	45	5	10	51	7	9

$GF[13^2]$, $i^2 \equiv i + 11$, modulo 13. $i^\lambda = \alpha i + \beta$.

SECOND TABLE.—*Continued.*

λ	α	β	λ	α	β	λ	α	β	λ	α	β
75	7	10	160	9	1	61	10	5	17	11	9
22	7	11	133	9	2	142	10	6	130	11	10
156	7	12	148	9	3	59	10	7	40	11	11
43	8	0	41	9	4	161	10	8	19	11	12
107	8	1	31	9	5	11	10	9	85	12	0
81	8	2	9	9	6	82	10	10	13	12	1
129	8	3	144	9	7	37	10	11	86	12	2
131	8	4	114	9	8	8	10	12	123	12	3
139	8	5	54	9	9	99	11	0	120	12	4
38	8	6	108	9	10	119	11	1	116	12	5
78	8	7	33	9	11	27	11	2	5	12	6
152	8	8	151	9	12	163	11	3	105	12	7
63	8	9	141	10	0	100	11	4	149	12	8
44	8	10	136	10	1	94	11	5	80	12	9
90	8	11	4	10	2	137	11	6	132	12	10
74	8	12	69	10	3	146	11	7	3	12	11
113	9	0	20	10	4	134	11	8	26	12	12

NOTES.

THE July number (volume 6, number 3) of the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY contains the following papers: "Sur les lignes géodésiques des surfaces convexes," by H. POINCARÉ; "The classification of quadrics," by T. J. I'A. BROMWICH; "On differential invariants," by J. E. WRIGHT; "Groups of order p^m, which contain cyclic subgroups of order p^{m-3}," by L. I. NEIKIRK; "On the invariant subgroups of prime index," by G. A. MILLER; "On a general method for treating transmitted motions and its application to indirect perturbations," by E. W. BROWN; "On hypercomplex number systems," by L. E. DICKSON; "A theorem on finite algebras," by J. H. MACLAGAN-WEDDERBURN; "The relation of the principles of logic to the foundations of geometry," by J. ROYCE; "On multiple integrals," by J. PIERPONT.

THE July number (volume 27, number 3) of the *American Journal of Mathematics* contains: "Deduction of the power series representing a function from special values of the latter," by G. W. HILL; "On the definition of reducible hypercomplex number systems," by S. EPSTEEN and H. B. LEONARD;

"Quintic curves for which $p = 1$," by P. FIELD; "Classification of the surfaces of singularities of the quadratic spherical complex," by C. L. E. MOORE; "Subgroups of order a power of p in the general and special m-ary linear homogeneous groups in the $GF[p^n]$," by L. E. DICKSON.

THE concluding (July) number of volume 6 of the *Annals of Mathematics* contains: "On the real elements of certain classes of geometrical configurations," by L. E. DICKSON; "The continuum as a type of order; an exposition of the modern theory, I–IV," by E. V. HUNTINGTON; "On integrating factors," by P. SAUREL; "Concerning series of analytic functions," by M. B. PORTER.

THE conference of associations of teachers of mathematics and physics, held at Asbury Park on July 5 in connection with the meeting of the National educational association was attended by thirty-seven delegates representing nearly all the larger teachers' associations in this field. Professor THOMAS S. FISKE was elected chairman, and Dr. ARTHUR SCHULTZE secretary of the conference. It was decided to organize a national society of teachers of mathematics and science and a committee was appointed to arrange the details of the organization.

WHILE in this country last autumn Professor G. Darboux became interested in the proper reporting of articles appearing in American mathematical publications for the *Bulletin des sciences mathématiques*. Arrangements have been made by which the *American Journal of Mathematics*, the *Transactions of the American Mathematical Society* and the *Annals of Mathematics* will be regularly reviewed in Darboux's *Bulletin* by Professor J. W. YOUNG, of Princeton University, Professor E. R. HEDRICK, of the University of Missouri, and Dr. C. M. MASON, of Yale University, respectively. Articles of an original nature appearing in the BULLETIN will also be mentioned. Authors of articles in these publications will render valuable service by furnishing abstracts of their articles to the respective reviewers.

THE following books will shortly appear from the G. J. Göschen'sche publishing house, Leipzig: "Auslese aus meiner Unterrichts- und Vorlesungspraxis," by Dr. H. SCHUBERT; "Vorlesungen über photographische Optik," by Dr. A. GLEICHEN; "Zwölf Vorlesungen über die Natur des Lichtes,"

by Dr. J. CLASSEN; "Mehrdimensionale Geometrie, II. die Polytope (Sammlung Schubert, number 36)," by Professor P. H. SCHOUTE.

UNIVERSITY OF COLORADO. — The following advanced courses in mathematics are announced for the academic year 1905–1906: By Professor I. M. DELONG: Theory of equations, five hours; Differential equations, three hours. — By Dr. S. EPSTEEN: Linear algebra, three hours; Least squares and theory of errors, three hours; Mathematics of investments (including insurance), three hours. — By Mr. H. B. LEONARD: Theory of Functions, three hours. — By Miss R. L. CARSTENS: History of mathematics, two hours.

THE various foreign universities below offer courses in mathematics for the winter semester of 1905–1906 as follows:

UNIVERSITY OF GÖTTINGEN. — By Professor F. KLEIN: Projective geometry, including non-euclidean geometry, four hours; Seminar, two hours. — By Professor D. HILBERT: Introduction to partial differential equations, two hours; Mechanics, four hours; Seminar, two hours. — By Professor H. MINKOWSKI: Algebraic curves and surfaces, two hours; Theory of numbers, four hours; Seminar (with Professor Hilbert), two hours. — By Professor C. RUNGE: Differential and integral calculus, II, four hours; Graphical methods in mechanics, three hours. — By Professor K. SCHWARZSCHILD: Celestial mechanics, II, three hours. — By Professor W. VOIGT: Elasticity and acoustics, four hours; Electro-optics, two hours. — By Dr. E. ZERMELO: Functions of real variables, three hours. — By Dr. M. ABRAHAM: Maxwell's theory and Hertzian waves, three hours. — By Dr. O. BLUMENTHAL: Theory of surfaces, three hours. — By Dr. C. HERGLOTZ: Minimum surfaces, two hours. — By Dr. C. CARATHÉODORY: Continuous groups, three hours.

UNIVERSITY OF HALLE. — By Professor G. CANTOR: Theory of analytic functions, four hours; Seminar, two hours. — By Professor A. WANGERIN: Applications of elliptic functions, two hours; Theory of potential and spherical harmonics, four hours; Seminar, two hours. — By Professor A. GUTZMER: Analytic mechanics, four hours; Theory and applications of determinants, two hours; Seminar, two hours. — By Professor V. EBERHARD: Integral calculus, four hours; with

exercises, one hour. — By Dr. F. BERNSTEIN: Analytic geometry of space, two hours; Theory of differential equations, two hours.

UNIVERSITY OF INNSBRUCK. — By Professor O. STOLZ: Differential and integral calculus of real variables, three hours; Concept of complex numbers as introduction to the theory of functions, two hours; Seminar, one hour. — By Professor K. ZINDLER: Analytic geometry of two and three dimensions, with exercises, seven hours. — By Professor K. MENGER: Descriptive geometry, three hours.

UNIVERSITY OF JENA. — By Professor J. THOMAE: Elementary theory of functions, four hours; Analytic geometry of space, four hours; Seminar, two hours. — By ——— (not yet appointed): Integral calculus with exercises, five hours; Theory and applications of determinants, two hours; Elements of the theory of numbers, two hours. — By Professor G. FREGE: Analytic mechanics, four hours; Seminar, two hours.

UNIVERSITY OF KÖNIGSBERG. — By Professor W. F. MEYER: Theory of algebraic equations, four hours; Seminar, one hour. — By Professor A. SCHOENFLIES: Integral calculus, four hours; with exercises, two hours. — By Professor L. SAALSCHÜTZ: Introduction to algebraic analysis, four hours; Geometry of space, three hours; with exercises, one hour. — By Dr. I. COHN: Theory of potential, three hours.

UNIVERSITY OF MUNICH. — By Professor G. BAUER: Seminar, two hours. — By Professor F. LINDEMANN: Differential calculus, five hours; Analytic mechanics, four hours; Seminar, one and one half hours. — By Professor A. VOSS: Plane analytic geometry, five hours; Algebraic surfaces, four hours; Seminar, two hours. — By Professor A. PRINGSHEIM: Elliptic functions, four hours; Continued fractions, two hours. — By Professor K. DOEHLEMANN: Descriptive geometry, I, five hours; with exercises, three hours; Synthetic geometry, five hours; with exercises, one hour. — By Professor E. v. WEBER: Algebra, four hours; Integral calculus with exercises, five hours. — By Dr. H. BRUNN: Theory of aggregates, four hours.

OXFORD UNIVERSITY. *Michaelmas term.* — By Professor W. ESSON: Analytic geometry of plane curves, two hours;

Synthetic geometry of plane curves, one hour. — By Professor E. B. ELLIOTT: Sequences and series, two hours; Elementary theory of numbers, one hour. — By Professor A. E. H. LOVE: Mathematical theory of electricity and magnetism, three hours. — By Professor H. H. Turner: Elementary mathematical astronomy, two hours. — By Mr. C. E. HASELFOOT: Theory of equations, one hour. — By Mr. C. LEUDESDORF: Projective geometry, three hours. — By Mr. A. E. JOLLIFFE: Analytic geometry, two hours. — By Mr. J. W. RUSSELL: Differential calculus, two hours. — By Mr. B. F. McNEILE: Curve tracing, one hour. — By Mr. A. L. PEDDER: Problems in pure mathematics, one hour. — By Mr. C. H. SAMPSON: Solid geometry, two hours. — By Mr. J. E. CAMPBELL: Differential equations, two hours. — By Mr. C. H. THOMPSON: Integral calculus, two hours.

UNIVERSITY OF WÜRZBURG. — By Professor F. PRYM: Differential calculus with introduction to higher analysis, four hours; Plane analytic geometry, I, four hours; Proseminar, two hours; Seminar, two hours. — By Professor E. SELLING: Theory of algebraic equations, four hours; Analytic mechanics, four hours. — By Professor G. ROST: Theory of partial differential equations, four hours; Theory of invariants, four hours; Analytic geometry of space, four hours; Proseminar, six hours; Seminar, two hours.

THE Academy of sciences of Berlin held its Leibniz session on June 29. The Steiner prize was not awarded, but the sum of six thousand marks was set apart in recognition of the investigations of the late Professor GUIDO HAUCK. For the year 1910 the Academy announces the same problem for the third time for the Steiner prize:

"To solve completely any important, hitherto unsolved problem relating to the theory of curved surfaces, taking into account, so far as possible, the methods and principles evolved by Steiner. It is required that sufficient analytic explanations shall accompany the geometric investigations to verify the correctness and completeness of the solution. Without wishing to limit the choice of subject, the academy takes the opportunity to call attention to the special problems to which Steiner has referred in his general remark at the end of his second paper on maximum and minimum in figures in a plane, on a sphere, and in space."

Competing memoirs should be written in German, French, English, Italian or Latin, and must be submitted before December 31, 1909, under the usual conditions. The result will be announced at the Leibniz session of 1910.

THE London mathematical society has awarded the De Morgan prize for the present year to Dr. H. F. BAKER, for his work in analysis.

THE Hungarian academy of sciences has established a new prize, known as the Bolyai prize, in commemoration of JOHN and WOLFGANG BOLYAI. It consists of a medal and a sum of ten thousand crowns. It will be awarded this fall to the author of the best mathematical production during the last five years, and at intervals of five years thereafter. The committee of award consists of Professors Darboux, Klein, Kõnig and Rádos; it will meet in Budapest in October.

PROFESSOR W. WIRTINGER has been elected a member of the academy of sciences of Vienna.

PROFESSOR J. FRANEL, of the technical school at Zürich, has been chosen director of that institution.

MR. E. T. WHITTAKER, of Trinity College, Cambridge, has been appointed university lecturer in mathematics.

PROFESSOR J. E. PAPPERITZ, of the mining academy at Freiberg, has been elected rector of the academy for the present academic year.

PROFESSOR C. NEUMANN, of the University of Leipzig, has been decorated with the red cross of the second class.

PROFESSOR W. VOIGT, of the University of Göttingen, has been decorated with the order of the red cross of the third class.

DR. H. C. TIMERDING has been appointed professor of applied mathematics at the University of Strassburg.

PROFESSOR A. GUTZMER, of the University of Jena, has been appointed professor of mathematics at the University of Halle.

PROFESSOR L. HEFFTER, of the technical school at Aachen, has been appointed professor of mathematics at the University of Kiel.

DR. O. BLUMENTHAL, of the University of Göttingen, has been appointed professor of mathematics at the technical school at Aachen.

DR. L. FEJER has been appointed docent in analysis and mechanics at the University of Clausenburg.

THE following changes have been made at the University of Illinois: Professor E. J. TOWNSEND has been made full professor of mathematics and acting dean of the college of science; Professor S. E. SLOCUM, of the University of Cincinnati, has been appointed to an assistant professorship of mathematics; Dr. L. NEIKIRK, of the University of Pennsylvania, has been appointed instructor in mathematics, and Mr. W. J. RISLEY, of the Armour Institute, and M. C. EMMENS, of Albion College, have been made assistants in mathematics. Professor A. G. HALL has been appointed professor of mathematics in Miami University, and Mr. A. H. WILSON has been appointed associate professor of mathematics at the Polytechnic Institute of Alabama.

THE College entrance examination board has appointed as examiners in mathematics for 1905–1906 Professor F. N. COLE, Professor H. S. WHITE, and Dr. ARTHUR SCHULTZE.

DR. D. R. CURTISS, of Yale University, has been appointed to an assistant professorship of mathematics at Northwestern University.

AT the University of Missouri, Dr. O. D. KELLOGG of Princeton University has been appointed to an assistant professorship of mathematics, and Mr. R. L. BORGER, of the University of Florida, to an instructorship in mathematics.

AT Northwestern University, Professor THOMAS F. HOLGATE has been appointed Noyes professor of pure mathematics; Mr. R. E. WILSON has returned from Europe to resume his work as instructor in mathematics; Dr. J. C. MOREHEAD has been appointed instructor in mathematics.

AT Princeton University Dr. L. P. EISENHART and Dr. W. GILLESPIE have been promoted to assistant professorships of mathematics. Dr. O. VEBLEN, of the University of Chicago, and Dr. J. W. YOUNG, of Northwestern University, have been appointed assistant professors of mathematics, and Dr. O. L. UNDERHILL instructor in mathematics.

DR. L. I. HEWES has been appointed instructor in mathematics at Yale University.

DR. R. B. ALLEN has been appointed lecturer in mathematics at Clark University during Professor Taber's absence.

DR. S. EPSTEEN, of the University of Chicago, has been appointed instructor in applied mathematics at the University of Colorado.

PROFESSOR A. EMCH, of the University of Colorado, has been appointed professor of mathematics at the cantonal college of Solothurn, Switzerland.

DR. O. P. AKERS has been appointed assistant professor of mathematics at Allegheny College, Meadville, Pennsylvania.

RECENT catalogues of second-hand mathematical books: Macmillan and Bowes, 1 Trinity St., Cambridge, England, catalogue no. 305, 5300 mathematical pamphlets; catalogue no. 302, from the libraries of Pendlebury, Spottiswoode and Whitehead, about 1400 titles in mathematics. — Martin Boas, Karltrasse 25, Berlin, catalogue no. 49, 56 titles in mathematics and geodesy. — H. R. Mecklenburg, 36 Klosterstrasse, Berlin, 750 titles in mathematics and geodesy.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

AMODEO (F.). Lezioni di geometria proiettiva dettata nella r. Università di Napoli. 3a edizione. Napoli, Pierro, 1905. 8vo. 14 + 456 pp.

BACHMANN (P.). Zahlentheorie. Versuch einer Gesamtdarstellung dieser Wissenschaft in ihren Hauptteilen. Band V: Die allgemeine Arithmetik der Zahlenkörper. Leipzig, Teubner, 1905. 8vo. 22 + 548 pp. M. 16.00

BERKHAU (G.). Zur projektiven Behandlung der Dreiecksgeometrie. (Diss.) Königsberg, 1905. 8vo. 31 pp.

BOCCHETTA (G.). Principi di geometria metrica e differenziale negli iperspazi. Melfi, Grieco, 1905. 8vo. 26 pp.

CAIN (W.). A brief course in the calculus. New York, Van Nostrand, 1905. 8vo. 10 + 280 pp. Cloth. (Van Nostrand series of text-books.) $1.75

COMBÉBIAC (G.). Calcul des triquaternions; nouvelle analyse géométrique. (Thèse.) Paris, Gauthier-Villars, 1905. 4to. 122 pp.

CREFCOEUR (A. J. M.). Cours d'analyse. Partie II: Calcul des différences et calcul des variations. Paris, 1905. 8vo. Fr. 3.50

DÖLLE (R.). Orthogonale Invarianten der Circularkurven dritter Ordnung. (Diss.) Jena, 1905. 8vo. 38 pp.

EPSTEEN (S.) and LEONARD (H. B.). On the definition of reducible hypercomplex number systems. 4to. (*American Journal of Mathematics* 27, pp. 217–242).

FIELD (P.). Quintic curves for which $p=1$. 4to. (*American Journal of Mathematics* 27, pp. 243–247, 10 plates.)

FUHRMANN (W.). Aufgaben aus der analytischen Geometrie. Königsberg, 1905. 4to. 28 pp.

GERLICH (P.). Ueber den Tangentenkomplex der Fläche zweiten Grades. (Diss.) Breslau, 1905. 8vo. 59 pp.

GIUDICE (M. DEL). Sulle equazioni algebriche di grado $2n$ la cui risoluzione si riconduce alla risoluzione, mediante equazioni di grado n, a quella di n equazioni quadratiche. Nota I e II. Napoli, Trani, 1904. 4to. 16 + 16 pp.

HALSTED (G. B.). See HILBERT (D.).

HERMITE (C.). Oeuvres, publiées sous les auspices de l'Académie des sciences par E. Picard. Vol. I. Paris, Gauthier-Villars, 1905. 8vo. 40 + 498 pp., portrait. Fr. 18.00

HILBERT (D.). On the foundations of logic and arithmetic. Translated by G. B. Halsted. 8vo. (*The Monist*, July, 1905, pp. 338–352.)

KRAZER (A.). See VERHANDLUNGEN.

LANGE (D. DE). Eenige beschouwingen over enkel- en meervoudige ombilicaalpunten en den loop der krumtelijnen in hunne nabijheid. Amsterdam, 1904. 8vo. 8 + 82 pp. M. 3.00

LEONARD (H. B.). See EPSTEEN (S.).

LINDOW (M.). See SOHNCKE (L. A.).

MONTESSUS DE BALLORE (R. DE). Les fractions continues algébriques. (Thèse.) Paris, Hermann, 1905. 4to. 85 pp.

MÜLLER (C. H.). Studien zur Geschichte der Mathematik, insbesondere des mathematischen Unterrichts an der Universität Göttingen im 18. Jahrhundert. Mit einer Einleitung: Ueber Charakter und Umfang historischer Forschung in der Mathematik. (Diss., Göttingen.) Leipzig, Teubner, 1904. 8vo. 94 pp.

NEUMANN (E. R.). Studien über die Methoden von C. Neumann und G. Robin zur Lösung der beiden Randwertaufgaben der Potentialtheorie. Leipzig, Teubner, 1905. 8vo. 23 + 194 pp. (Preisschriften, gekrönt und herausgegeben von der Fürstlich Jablonowskischen Gesellschaft zu Leipzig, 37, No. 15 der mathematisch-naturwissenschaftlichen Sektion.) M. 10.00

PICARD (E.). See HERMITE (C.).

QUIDDE (W.). Ueber Gauss'sche Kreise auf Rotationsflächen. (Diss.) Kiel, 1905. 8vo. 85 pp.

SALVADORI (M.). Esposizione della teoria delle somme di Gauss e di alcuni teoremi di Eisenstein. (Tesi, Friburgo.) Pisa, 1905. 8vo. 116 pp.

SCHMALL (S. M.). A first course in analytical geometry, plane and solid, with numerous examples. New York, Van Nostrand, 1905. 8vo. 7 + 318 pp. Cloth. (Van Nostrand series of text-books.) $1.75

SOHNCKE (L. A.). Sammlung von Aufgaben aus der Differential- und Integralrechnung. Teil II: Integralrechnung. Abteilung 1. 6te, verbesserte Auflage, bearbeitet und herausgegeben von M. Lindow. Jena, Schmidt, 1905. 8vo. 12 + 221 pp. M. 4.00

SOUCHON (A.). Sur l'intégration des équations générales de la dynamique analytique et sur les principales propriétés de leurs intégrales canoniques. Tours, Deslis, 1905. 8vo. 23 pp.

TABER (H.). On hypercomplex number systems. 4to. (*Transactions of the American Mathematical Society*, 5, pp. 509–548.)

——— The scalar functions of hypercomplex numbers. 8vo. (*Proceedings of the American Academy of Arts and Sciences*, 41, pp. 59–70.)

VERHANDLUNGEN des dritten internationalen Mathematiker-Kongresses in Heidelberg vom 8. bis 13. August 1904. Herausgegeben von dem Schriftführer des Kongresses A. Krazer. Mit einer Ansicht von Heidelberg in Heliogravüre und 2 Doppeltafeln. Leipzig, Teubner, 1905. 8vo. 10 + 756 pp. M. 18.00

VERONESE (G.). La geometria non archimedea; una questione di priorità. Roma, 1905. 8vo. (*Rendiconti della r. Accademia dei Lincei* 14, pp. 247–251.)

WIELEITNER (H.). Bibliographie der höheren algebraischen Kurven für den Zeitabschnitt von 1890–1904. Leipzig, 1904. 8vo. 53 pp. M. 1.50

WILDERVANCK (J. C.). De verschillende krommingen eener gewrongen kromme der vierdimensionale ruimte. Groningen, 1904. 8vo. 62 pp. M. 2.50

WILSON (E. B.). On products in additive fields. 8vo. (*Verhandlungen des III. Internationalen Mathematiker-Kongresses*, Heidelberg, 1904, pp. 202–215.)

II. ELEMENTARY MATHEMATICS.

AMODEO (F.). Equazioni di primo grado ad una incognita. Appendice all' Aritmetica particolare e generale estratta dagli Elementi di algebra dello stesso autore, per uso delle scuole normali e tecniche. Napoli, Pierro, 1904. 16mo. 31 pp. Fr. 0.30

ARBES (J.). Vierstellige Logarithmen-Tafeln zum Schulgebrauche. Wien, Tempsky, 1905. 8vo. 36 pp. Boards. M. 0.70

BACHET (C. G.). Problèmes plaisants et delectables qui se font par les nombres, par Claude-Gaspar Bachet, sieur de Méziriac. 4e édition, revue et simplifiée. Paris, Gauthier-Villars, 1905. 16mo. 170 pp. Fr. 3.50

BALL (W. W. R.). Mathematical recreations and essays. 4th edition. London, Macmillan, 1905. 12mo. 418 pp. Coth. 7s.

BARNES (A.). See CONSTERDINE (A.).

BORCHARDT (W. G.). Key to New trigonometry for schools. Part I. London, Bell, 1905. 12mo. Cloth. 5s.

BOURLET (C.). Précis d'algèbre, contenant cinq cent cinquante-sept exercices et problèmes rédigés conformément aux programmes du 31 mai 1902. Classes de seconde et première C et D. 2e édition. Paris, Hachette, 1905. 16mo. 393 pp. (Cours complet de mathématiques.) Fr. 2.50

CONSTERDINE (A.) and BARNES (A.). Rudiments of practical mathematics. London, Murray, 1905. 12mo. 348 pp. Cloth. (School Library.) 2s. 6d.

CORNEJO (H.). See GRAIÑO (F.).

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